Credible Scores

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Abstract

We study multidimensional cheap talk with simple language and aligned preferences. An expert communicates with a decision-maker using a score that aggregates a multidimensional state into a one-dimensional message. Expert and decision-maker share the same quadratic-loss utility function. We show that the use of simple language introduces strategic considerations. First, equilibrium payoffs may be lower than those achievable under commitment to a score. Additionally, any equilibrium score must be either linear or discrete. Finally, for normally distributed states, the set of equilibrium linear scores includes only the ex-ante best and worst linear scores.

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Give me a one-handed economist.

Harry Truman

1 Introduction

Experts often advise decision-makers who lack specialized knowledge. To reach their audience, experts have to adopt a simple language. Scientists, for instance, publish recommendations such as "5 a day" and rely on coarse metrics such as carbon footprints or nutritional labels. Similarly, product reviewers employ simplified quality indicators, such as star ratings or letter grades. In this study, we examine the strategic incentives that arise when experts communicate using simplified language, which we refer to as "scores".

We focus on *credible* scores, i.e., scores that are equilibrium strategies. A score is credible if, once the expert observes the relevant features of the state of the world, he has no incentive to misreport the score. To isolate the effect of strategic incentives, we focus on settings in which the sender and receiver share identical preferences. If the expert could use a language as rich as the object described, the alignment of preferences would make credibility a vacuous constraint: when language is sufficiently rich, revealing every relevant aspect of the object under consideration is both optimal and credible. When experts communicate using simplified language, the nature of optimal and credible communication is not immediately clear.

We explore communication via scores in a multi-dimensional cheap talk game with aligned preferences. A sender knows a two-dimensional state of the world. A receiver takes a two-dimensional action to minimize a quadratic loss function.[1] Sender and receiver share the same payoffs. A score is a mapping from the state space to a real number that satisfies a property we dub "Intermediate Value Property". Essentially, we require that small changes in the state of the world can only cause small changes in the score. The property captures the idea that the score must represent the underlying physical reality of the state space. All continuous scores satisfy the property. If the score has a countable image, e.g., a five-star rating, a marginal change in the state cannot make the score change by more than one star.

[1]The model could also describe a sender who addresses two different audiences with the same message.

We say that a score is credible if there is a Perfect Bayesian Equilibrium where the sender maps states of the world into messages according to the score.[2]

Our definition of scores captures a notion of "simple language". For instance, scores require the language to be coarser than the state: bijections between the state space and the real line do not satisfy the Intermediate Value Property. At the same time, our definition is flexible: it does not impose monotonicity or other functional form assumptions and accommodates both discrete and continuous images of the mapping from states to messages.

Our first result shows that communicating through scores can lead to welfare losses due to strategic frictions: in some situations, no ex-ante optimal score is credible. This is possible because the sender can deviate from the optimal score to a strategy that is not a score, once the receiver's expectations are set. Whenever such a deviation is profitable, commitment has value.

We then characterize the shape of credible scores when the state space is $\mathbb{R}^2$. We show that credibility imposes two types of restrictions on the score. First, it imposes functional form restrictions: any credible score is either linear or is a discrete coarsening of a linear score. Second, in many cases, no linear score is credible and a credible score must be discrete. This shows that the sender needs to use a coarser language to maintain the credibility of the score – for example a five-star rating instead of a continuous one.

In some instances, credible linear scores exist. When the state is normally distributed, we show that there are exactly two credible linear scores. These correspond to the ex-ante best and worst linear scores. One score positively correlates the actions across dimensions, and the other one negatively correlates them. The optimality of each score depends on the correlation between the two dimensions of the state of the world. This result shows that some scores can have poor welfare properties while still being credible.

1.1 Related Literature

We introduce a new notion of simple language in cheap-talk models by requiring the sender to use a score, an aggregator of a multidimensional state, in equilibrium. Our definition of score rules out bijections while being flexible enough to allow for discrete or continuous scores.

[2]Note that we propose here an equilibrium selection criterion for simple languages, not a restriction on the set of strategies. In other words, the sender is allowed to deviate to any strategy.

We show that modeling simpler language through equilibrium properties introduces strategic considerations and imposes additional constraints on communication. Closest to our paper is the literature studying cheap talk models with aligned preferences and some form of language limitation.[3] Jäger et al. (2011) study a similar model where the sender is constrained to use a finite number of messages. They establish that the ex-ante optimal strategy is an equilibrium and study the stability of the equilibrium. In Blume and Board (2013) and Blume (2018) uncertainty about the language used can impede communication. These three papers find that optimal strategies are equilibrium strategies.[4] We take a different approach from these papers by requiring simplicity to be an equilibrium property instead of a constraint on the strategy space itself. In particular, the optimal scores are not necessarily equilibrium strategies and therefore strategic frictions impose constraints on communication beyond the properties of scores.

We also relate to the literature on multidimensional cheap talk. This literature has shown that multiple dimensions can be useful for information revelation, e.g., Battaglini (2002), Chakraborty and Harbaugh (2007) and Chakraborty and Harbaugh (2010). In this strand of the literature, the contribution closest to ours is Levy and Razin (2007), who show that correlation across dimensions can limit communication by creating informational spillovers across dimensions. Similar mechanisms are at play in our paper as the sender needs to balance how the score, a one dimensional object, reveals information across both dimensions.

Finally, there is a strand of the literature in information design where the amount of information transmitted is limited. In Gentzkow and Kamenica (2014), the limitation comes from the cost of designing the experiment, while in Bloedel and Segal (2021) it comes from the information-processing cost faced by the receiver. When considering optimal scores, we impose a restriction directly on the shape of the information structure, by limiting the sender to select among scores. In this way we are closer to Le Treust and Tomala (2019) and Aybas and Turkel (2024), who consider exogenous constraints on the capacity or cardinality of the message space.

[3] This literature, like us, looks at the consequences of language limitations, not its causes. On the latter topic see Lipman (2003) and Lipman (2025).

[4] Similarly, Lipman (2025) uses the fact that optimal strategies are equilibrium strategies to show that there is always an equilibrium in pure strategy when preferences are aligned in cheap-talk games with a possibly constrained set of messages.

2 Model

There are two players: a sender and a receiver. The sender has private information about a two-dimensional state of the world, $\theta = (\theta_1, \theta_2) \in \Theta \subseteq \mathbb{R}^2$, whose distribution admits a density function f if the state is infinite. Otherwise, f denotes the probability mass function. We assume that the variance of θ is finite. When there is some ambiguity, we use $\tilde{\theta}$ to denote the random variable with realization θ. The receiver takes two actions represented by $a = (a_1, a_2) \in \mathbb{R}^2$. Before the receiver takes action, the sender sends a cheap-talk message $m \in \mathbb{R}$. Sender and receiver share the same payoff function

$$u(a, \theta) = -\phi(a_1 - \theta_1)^2 - (a_2 - \theta_2)^2,$$

with $\phi > 0$. Both players want each action to match the state. The parameter ϕ determines the dimension along which the loss from mismatch is the largest. Let $\mu : \Theta \to \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}^2$ denote pure strategies of the sender and the receiver. Also, for any $m \in \mathbb{R}$ and $i = 1, 2$, let $\alpha_i(m)$ denote the $i-$th element of $\alpha(m)$.

We are interested in a class of Perfect Bayesian Equilibria that we define in the next section.

An example of this setting is an expert giving advice to a government that needs to design a multidimensional policy. For example, promoting a healthy diet among different subpopulations, choosing tax levels for different groups or taking multiple investment decisions in some technology.

Our model is also equivalent to a model with two receivers, each taking a one-dimensional action. Each receiver minimizes a one-dimensional quadratic loss function and the sender maximizes a weighted sum of the receivers' payoffs. An example here could be an expert directly promoting a healthy diet among different subpopulations.

2.1 Scores

A *score* s is a non-constant function from Θ to $\mathbb{R}$ that satisfies the following property:

> Intermediate Value Property (IVP): for any $\theta, \theta' \in \Theta$ such that $s(\theta) > s(\theta')$ and any $m \in [s(\theta'), s(\theta)] \cap s(\Theta)$, there is a $\theta'' \in s^{-1}(m)$ such that $\theta \wedge \theta' \leq \theta'' \leq \theta \vee \theta'$.[5]

[5]Here, $\wedge$ is the component-wise minimum and $\vee$ is the component-wise maximum: $\theta \wedge \theta' =$

We discuss this definition in more detail below.

The set of scores is denoted by $\mathcal{S}$, and we refer to the typical realization of s as m. We say that a score is *optimal* if it solves the following maximization problem:

$$\max_{s \in \mathcal{S}} \mathbb{E}[-\phi(\alpha_1(s(\theta)) - \theta_1)^2 - (\alpha_2(s(\theta)) - \theta_2)^2]$$
$$\text{s.t.} \quad \alpha(m) = \mathbb{E}[\theta|m], \quad \forall m \in s(\Theta). \tag{BR}$$

A score is thus optimal if it maximizes the expected payoff among scores, given that the receiver best responds.

We say that a score $s : \Theta \to \mathbb{R}$ is *credible* if there is a Perfect Bayesian equilibrium (PBE) such that $\mu(\theta) = s(\theta)$ for all θ. A score is thus credible if and only if there is α that satisfies (BR) and $\forall m, m' \in s(\Theta)$ and $\forall \theta \in \Theta$:

$$s(\theta) = m \Rightarrow -\phi(\alpha_1(m) - \theta_1)^2 - (\alpha_2(m) - \theta_2)^2 \geq -\phi(\alpha_1(m') - \theta_1)^2 - (\alpha_2(m') - \theta_2)^2. \tag{IC}$$

We note that a credible score always exists.

Proposition 1. *A credible score exists.*

The proof is in Section A. We show existence of a credible score by showing that there always exists a PBE with two messages in the support of the sender's strategy. Because a non-constant strategy with two messages satisfies all the properties of a score, a credible score exists.

A score aggregates the two-dimensional state of the world into a one-dimensional object. The Intermediate Value Property ensures that the score is a well-behaved aggregator of the two-dimensional state of the world. Its economic interpretation is that it imposes a weak form of continuity: small changes in the state correspond to small changes in the score. We regard this property as a minimal requirement that the score must represent the underlying physical reality of the state space. All continuous mappings from Θ to $\mathbb{R}$ satisfy the property. For discrete mappings instead, the property requires that a minimal increment in the state changes the score by at most one grade. On a mathematical level, the property also rules out bijections between $\mathbb{R}^2$ and $\mathbb{R}$, in line with our original motivation.[6]

$(\min\{\theta_1, \theta_1'\}, \min\{\theta_2, \theta_2'\})$ and $\theta \vee \theta' = (\max\{\theta_1, \theta_1'\}, \max\{\theta_2, \theta_2'\})$.

[6]Similarly, one can also show that the IVP rules out bijections for scores of the form $s : \{1, ..., n\}^2 \to \mathbb{R}$.

Lemma 1. *A score $s : \mathbb{R}^2 \to \mathbb{R}$ is not a bijection.*

Proof. If $|s(\Theta)| \leq 2$, then s cannot be a bijection. Take some messages $m, m_1, m_2 \in s(\Theta)$ with $m_1 < m < m_2$. Take $\theta, \theta^1, \theta^2$ such that $s(\theta) = m$, $s(\theta^1) = m_1$ and $s(\theta^2) = m_2$.

We can draw a curve from θ^1 to θ^2 consisting of straight vertical and horizontal segments such that this curve does not intersect with θ. At least one of these segments has end points, denoted θ' and θ'', such that $s(\theta') \leq m \leq s(\theta'')$. By the IVP, there must be θ''' on that segment such that $s(\theta''') = m$. $\square$

Scores can exist if the state space is finite, e.g. $\Theta = \{0, 1\}^2$. Figure 1 shows 4 different scores for this space; in the figure, dots in the same area represent states to which the score assigns the same signal.

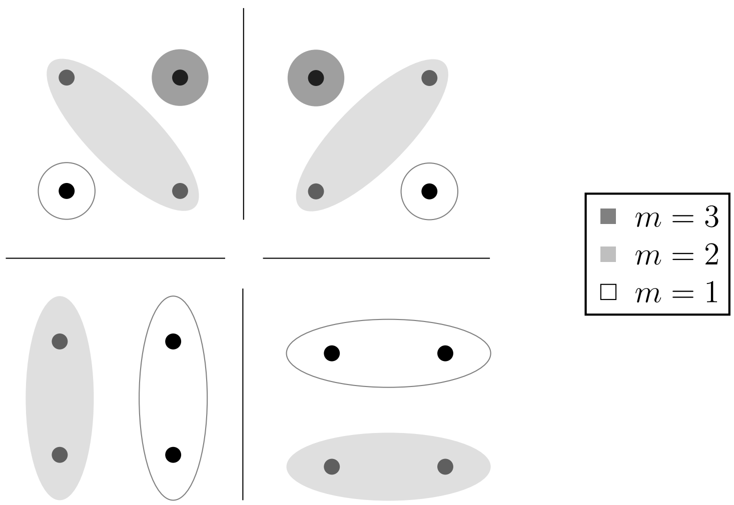

Figure 1: Examples of scores for $\Theta = \{0, 1\}^2$

When the space is infinite, e.g. $\Theta = \mathbb{R}^2$, scores can have finite images, e.g., five-star ratings, or infinite ones:

- $s(\theta) = \beta_0 + \beta_1\theta_1 + \beta_2\theta_2$;
- $s(\theta) = \begin{cases} 1 & \text{if } \beta_1\theta_1 + \beta_2\theta_2 \geq c, \\ 0 & \text{otherwise.} \end{cases}$;
- $s(\theta) = \sqrt{(\theta_1 - c_1)^2 + (\theta_2 - c_2)^2}$.

These examples show that scores can take many different forms. In particular, they can be continuous functions or take discrete values. The score does not need not be increasing or decreasing in any dimension. The last example shows a score that measures the distance between the state and a point (c_1, c_2) on the plane. If the state θ represents political positions along two dimensions, this score can be interpreted as a measure of extremism where (c_1, c_2) would be the political center.

3 Analysis

3.1 Value of Commitment

We first argue that commitment has value, i.e., that it can be the case that none of the optimal scores is credible. We make our argument with an example. Let $\phi = 1$, the state take values $\Theta = \{0, 1\}^2$ and, for simplicity, let $f(\theta) \neq f(\theta')$ for any two states $\theta \neq \theta'$. Let scores s_d and s_D be as shown, respectively, in the top left and top right panels of Figure 1. Score s_d assigns the same message to states $(0, 1)$ and $(1, 0)$ while assigning unique messages to the other states. Score s_D instead assigns the same message to states $(0, 0)$ and $(1, 1)$ while assigning unique messages to the other states. Up to an inconsequential relabeling of the messages, the optimal score is either s_d or s_D.

Remark 1. *The optimal score is either s_d or s_D. Score s_d is optimal if:*

$$\frac{f(0,0)f(1,1)}{f(0,0) + f(1,1)} \geq \frac{f(0,1)f(1,0)}{f(0,1) + f(0,1)}; \tag{1}$$

if the condition holds with a reversed inequality, score s_D is optimal.

The proof of this and the next remark are in Section B. The optimal score is credible if and only if the prior probabilities of the two states associated with the same message are not too different.

Remark 2. *Suppose the optimal score assigns the same signal to states θ and θ'. The optimal score is credible if and only if*

$$\frac{f(\theta)}{f(\theta')} \in \left[\sqrt{2} - 1, \frac{1}{\sqrt{2} - 1}\right].$$

The intuition is as follows. Suppose condition (1) holds strictly, so that s_d is the unique optimal score. Score s_d is shown on the left-hand side of Figure 2. Suppose also that $\frac{f(0,1)}{f(1,0)} > \frac{1}{\sqrt{2}-1}$, so that the posterior associated with $m = 2$ is "close" to $(0, 1)$ and "far" from $(1, 0)$. In fact, the posterior is so far from $(1, 0)$ that the score is not credible: if the receiver expects the sender to communicate according to the score, i.e., $\mu(\theta) = s_d(\theta)$ for all θ, then the sender has a profitable deviation upon observing state $(1, 0)$. Figure 2 shows one such deviation, which involves message $\mu(1, 0) = 3$ instead of $\mu(1, 0) = 2$.

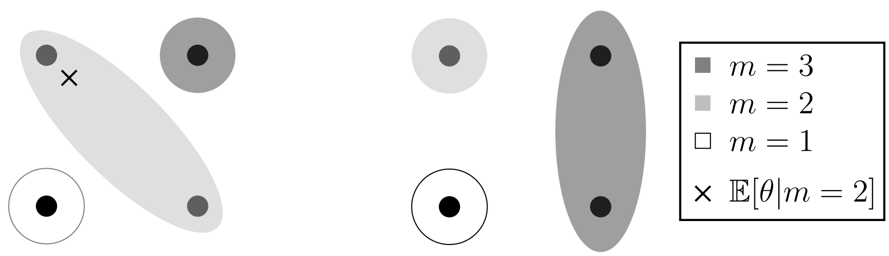

Figure 2: Left: Strategy $\mu = s_d$. Right: Profitable Deviation from $\mu = s_d$.

The deviation leads to a strategy that violates the IVP, as it "jumps" from $\mu(0, 0) = 1$ to $\mu(1, 0) = 3$. This strategy is not a score. In general, optimal scores need not be credible precisely because deviations to strategies that are not scores are possible.[7] This is in contrast with the rest of the literature that studies cheap talk models with aligned preferences (Jäger et al. (2011), Blume and Board (2013) and Blume (2018)) where the constraints on communication is on the message space directly and not on the properties of the equilibrium.

3.2 Infinite State Space

We characterize here credible scores when the state space is $\mathbb{R}^2$. We show that credible scores must satisfy specific properties that are imposed by the equilibrium conditions. We first introduce three definitions.

A score s is *linear* if there exists β_1 and β_2 such that, for any $\theta \in \mathbb{R}^2$,

$$s(\theta) = \beta_1\theta_1 + \beta_2\theta_2.$$

[7] Relatedly, in some cases, the players are better off if the sender only observed one state of the world (an example of such a case is available upon request). The intuition here is that ignorance reduces the set of potential deviations available to the sender.

A score s is *coarsely linear* if it has a discrete image $M \subseteq \mathbb{Z}$ and there exists β_1 and β_2 such that

$$s(\theta) = m \Leftrightarrow c_{m-1} < \beta_1\theta_1 + \beta_2\theta_2 \leq c_m,$$

with $-\infty \leq c_{m-1} < c_m \leq +\infty$ for any $\theta \in \mathbb{R}^2$.

Essentially, a coarsely linear score can be constructed by taking a linear score and partitioning its image into a countable number of intervals.

The scores s and s' are *equivalent* if $\mathbb{E}[\tilde{\theta}|s(\theta)] = \mathbb{E}[\tilde{\theta}|s'(\theta)]$ for all $\theta \in \Theta$.

We are now ready to state the result of this section.

Proposition 2. *Suppose $\Theta = \mathbb{R}^2$. Any credible score is equivalent to a linear or coarsely linear score.*

The proof is in Section C. To understand how we get Proposition 2, observe that given a belief about the sender's strategy, the receiver takes an action $\alpha(m) = \mathbb{E}[\theta|m]$. The sender's objective in state θ, given this belief, is to choose the message m' that minimizes the loss function:

$$\min_{m'} \left(\phi(\theta_1 - \alpha_1(m'))^2 + (\theta_2 - \alpha_2(m'))^2\right).$$

As the sender minimizes a weighted Euclidean distance, in any equilibrium the set of states indifferent between any two messages must be a line. Furthermore, the IVP requires that such indifference lines do not cross. These observations imply that every credible score with a discrete image must be coarsely linear. Linear scores can be seen as a limit case of coarsely linear scores. The rest of the proof shows that when the image of the score is not discrete, linear scores are the *only* scores compatible with credibility.

With a loss function that is not a quadratic one, the credibility of the score would impose other restrictions on the score's functional form. In light of this observation, Proposition 2 should not be interpreted as showing that linear strategies are special, but rather that credibility imposes functional form restrictions on communication.

A linear score is credible only if the receiver's expectations are linear in the score. When this condition is not met, all credible scores are coarsely linear. In these cases, the sender limits the information he transmits to maintain credibility. Indeed, every coarsely linear score can be improved upon by a score using more messages.

3.3 Normally Distributed State

There are important classes of distributions — such as the normal distribution — for which the conditional expectations given a linear score are linear. In this case, a credible linear score might exist. We now characterize the linear scores when the state is normally distributed.[8]

Let $\mathcal{S}_l = \{s : \mathbb{R}^2 \to \mathbb{R} : s \text{ is linear}\}$. We refer to a score as an *ex-ante best linear score* if it solves the problem:

$$\max_{s \in \mathcal{S}_l} \mathbb{E}[-\phi(\alpha_1(s(\theta)) - \theta_1)^2 - (\alpha_2(s(\theta)) - \theta_2)^2] \quad \text{s.t. } \alpha(m) = \mathbb{E}[\theta|m], \quad \forall m \in s(\Theta).$$

We instead refer to a score as an *ex-ante worse linear score* if it solves

$$\min_{s \in \mathcal{S}_l} \mathbb{E}[-\phi(\alpha_1(s(\theta)) - \theta_1)^2 - (\alpha_2(s(\theta)) - \theta_2)^2] \quad \text{s.t. } \alpha(m) = \mathbb{E}[\theta|m], \quad \forall m \in s(\Theta).$$

Let

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

be a covariance matrix and

$$\Phi = \begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}.$$

We identify a linear score $s(\theta) = \beta'\theta$ with the weights $\beta = (\beta_1, \beta_2)'$.[9]

Proposition 3. *Let $\theta \sim N(0, \Sigma)$. The set of credible linear scores are the eigenvectors of $\Sigma\Phi$. These are the ex-ante best and worst linear scores.*

The proof is in Section D. Proposition 3 shows that when the state is normally distributed, the best linear score is achievable in equilibrium. However, another linear equilibrium exists, corresponding to the worst possible linear score. The key idea underlying the proof is the following. A linear score β, with corresponding strategy α, is credible if the indifference curve of each type θ sending message m, $\{a \in \mathbb{R}^2 : u(a, \theta) = u(\alpha(m), \theta)\}$, is tangent to the curve $\{\alpha(m) : m \in \mathbb{R}\}$. We show that the linear scores β satisfying these tangency conditions are the eigenvectors of $\Sigma\Phi$. These eigenvectors, in turn, solve the first-order conditions of the ex-ante maximization problem.

[8] The next result remains valid under elliptical distributions — a broader class of distributions that also satisfy the linear conditional expectations property.

[9] We use the convention that when writing a vector as a matrix, it is a column vector.

The proof of Proposition 3 is general and can be extended to arbitrary dimensions of the state and action space. When the dimension is larger than two, the set of credible linear scores coincides with the set of stationary points of the ex-ante maximization problem. In the case of two dimensions, we can explicitly calculate the credible linear scores. Note that for any constant $c \neq 0$, two linear scores β' and β'' such that $\beta' = c\beta''$ induce the same distributions over actions. Therefore, any linear score is determined by the ratio β_1/β_2, if the ratio exists.

Corollary 1. *Suppose $\sigma_{12} \neq 0$. The credible linear scores, $\beta' = (\beta_1', \beta_2')$ and $\beta'' = (\beta_1'', \beta_2'')$, are determined by the ratios*

$$\frac{\beta_1'}{\beta_2'} = \frac{\phi\sigma_1^2 - \sigma_2^2 + \sqrt{(\phi\sigma_1^2 - \sigma_2^2)^2 + 4\phi\sigma_{12}^2}}{2\sigma_{12}} \neq 0,$$
$$\frac{\beta_1''}{\beta_2''} = \frac{\phi\sigma_1^2 - \sigma_2^2 - \sqrt{(\phi\sigma_1^2 - \sigma_2^2)^2 + 4\phi\sigma_{12}^2}}{2\sigma_{12}} \neq 0.$$

If $\sigma_{12} = 0$, then $\beta_2' = 0$ and $\beta_1'' = 0$, i.e., the credible scores fully reveal one dimension.[10]

The interpretation of a positive ratio β_1/β_2 is that a higher score is associated with a higher state: $\mathbb{E}[\theta_i|m]$ is increasing in m for $i = 1, 2$. If, for instance, the score rates a movie by considering its aesthetic quality, θ_1 and entertainment value, θ_2, then a higher score indicates that the movie has a higher expected value in both dimensions. If instead the ratio β_1/β_2 is negative, the score can be interpreted as a relative measure. A higher score indicates that a movie has a higher aesthetic value and less entertainment value.

When the correlation between the two dimensions is positive ($\sigma_{12} > 0$), the optimal linear scoring strategy satisfies $\beta_1/\beta_2 > 0$, which corresponds to inducing positively correlated actions by the receiver. When the correlation is negative, the best linear score is such that $\beta_1/\beta_2 < 0$, while the worst is such that $\beta_1/\beta_2 > 0$. It is worth noting that the worst score could be a natural candidate for a credible score. For example, if movie critics use a rating system where a higher rating indicates higher aesthetic or entertainment value but these two dimensions are negatively correlated, then the credible score has poor welfare properties.

Finally, Corollary 1 establishes that revealing one dimension is credible when the two states are uncorrelated. To understand this result, suppose that the sender uses a score that only reveals one dimension, say θ_1. Upon observing θ_1, the receiver will use the correlation between the two dimensions to make some inferences about θ_2. This reasoning from the receiver in-

[10]The proof of the corollary is immediate, therefore omitted.

troduces an incentive for the sender to lie about θ_1 to potentially correct the inference on θ_2. The intuition is that an appropriately chosen marginal change in the score induces a marginal loss of zero along the revealed dimension θ_1 and a positive marginal benefit along the other dimension. This information spillover is similar to the result in Levy and Razin (2007) who show that misalignment in one dimension can hinder communication in another dimension where receiver and sender have aligned preferences.

4 Conclusion

We model a cheap-talk game with aligned preferences where the sender is constrained to use a score in equilibrium. We show that this restriction introduces strategic frictions despite the aligned preferences. These frictions can create a wedge between optimal and credible scores. They also put structure on the shape of credible scores.

The multidimensionality of our model plays a key role for our results. In particular, if the state were one-dimensional, any optimal score would be credible. In a one-dimensional model, the score can be defined in multiple ways. Let $\Theta \subseteq \mathbb{R}$ and let the sender send messages in $M \subseteq \mathbb{R}$. A score is a function s that satisfies

1. $s : \Theta \to M$ and
2. s satisfies IVP.

If either $M = \mathbb{R}$ or $M = \{1, ..., n\}$ for some $n \in \mathbb{N}$, then any optimal score is credible. If $M = \mathbb{R}$, full revelation is possible so the optimal score is trivially credible. If $M = \{1, ..., n\}$, the result follows from the fact that for any given score and belief associated with it, the best profitable deviation is also a score. Therefore, if this deviation is profitable, then this score should have been optimal. This is the crucial difference with the two-dimensional case where a profitable deviation could be a strategy that is not a score.

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A Proof of Proposition 1

We first establish that there always exists an equilibrium with two messages.

Lemma 2. *There exist an equilibrium in which the sender chooses a strategy $\mu : \Theta \to \{1, 2\}$.*

Proof. As a first step, we establish that the function

$$v(\alpha^1, \alpha^2) = \int_\Theta \max\{u(\alpha^1, \theta), u(\alpha^2, \theta)\} dF$$

is continuous. To show this, we apply the dominated convergence theorem.

Take two converging sequences in $\mathbb{R}^2$, $(\alpha^{1,n}, \alpha^{2,n}) \to (\alpha^1, \alpha^2)$. Observe that

$$|\max\{u(\alpha^{1,n}, \theta), u(\alpha^{2,n}, \theta)\}| \leq \phi(\alpha_1^{1,n} - \theta_1)^2 + (\alpha_2^{1,n} - \theta_2)^2.$$

For any converging sequence in $\mathbb{R}^2$, $\alpha^n \to \alpha$, the function

$$\phi(\alpha_1^n - \theta_1)^2 + (\alpha_2^n - \theta_2)^2 = (\phi\theta_1^2 + \theta_2^2) - 2(\phi\theta_1\alpha_1^n + \theta_2\alpha_2^n) + \phi(\alpha_1^n)^2 + (\alpha_2^n)^2$$

is dominated by an integrable function. This is the case, because the sequence (α^n) converges, hence it is bounded and $\phi(\alpha_1^n)^2 + (\alpha_2^n)^2 \leq M$ for some $M > 0$. Similarly, by the Cauchy-Schwartz inequality, $|\phi\theta_1\alpha_1^n + \theta_2\alpha_2^n| \leq \sqrt{M}(\theta_1^2 + \theta_2^2)$. Therefore,

$$|\max\{u(\alpha^{1,n}, \theta), u(\alpha^{2,n}, \theta)\}| \leq \phi(\alpha_1^{1,n} - \theta_1)^2 + (\alpha_2^{1,n} - \theta_2)^2 \leq (\phi\theta_1^2 + \theta_2^2) + 2\sqrt{M}(\theta_1^2 + \theta_2^2) + M,$$

for some $M > 0$. Because the variance of θ is finite, the dominating function is integrable.

It is also clear that

$$\max\{u(\alpha^{1,n}, \theta), u(\alpha^{2,n}, \theta)\} \to \max\{u(\alpha^1, \theta), u(\alpha^2, \theta)\}, \text{ for each } \theta.$$

Therefore by the dominated convergence theorem,

$$\int_\Theta \max\{u(\alpha^{1,n}, \theta), u(\alpha^{2,n}, \theta)\} dF \to \int_\Theta \max\{u(\alpha^1, \theta), u(\alpha^2, \theta)\} dF,$$

and the function $v(\alpha^1, \alpha^2)$ is continuous.

As a second step, we establish that the following maximization problem has a solution:

$$\max_{\alpha^1, \alpha^2 \in \mathbb{R}^2} v(\alpha^1, \alpha^2) \tag{2}$$

The function $v(\alpha^1, \alpha^2)$ is bounded above by 0 and therefore a supremum exists, say v^*. Moreover, setting $\alpha^1 = \alpha^2 = \mathbb{E}[\theta]$ guarantees a payoff of $-\phi \operatorname{Var}[\theta_1] - \operatorname{Var}[\theta_2]$ and therefore $v^* \geq -\phi \operatorname{Var}[\theta_1] - \operatorname{Var}[\theta_2]$.

If $v^* = -\phi \operatorname{Var}[\theta_1] - \operatorname{Var}[\theta_2]$, then the supremum is attained by $\alpha^1 = \alpha^2 = \mathbb{E}[\theta]$ and therefore a maximum exists.

Suppose instead that $v^* > -\phi \operatorname{Var}[\theta_1] - \operatorname{Var}[\theta_2]$. Let $(\alpha^{1,n}, \alpha^{2,n})$ be a sequence such that $v(\alpha^{1,n}, \alpha^{2,n}) \to v^*$. We want to show that the sequence $(\alpha^{1,n}, \alpha^{2,n})$ is bounded.

Suppose it is not. If $\|\alpha^{k,n}\| \to \infty$, then $u(\alpha^{k,n}, \theta) \to -\infty$ for each θ.

If $\|\alpha^{k,n}\| \to \infty$ for both $k = 1, 2$, then $\max\{u(\alpha^{1,n}, \theta), u(\alpha^{2,n}, \theta)\} \to -\infty$ and therefore $v(\alpha^{1,n}, \alpha^{2,n}) \to -\infty$ and thus does not converge to v^*.

If $\|\alpha^{k,n}\| \to \infty$ for only one $k = 1, 2$, then $\alpha^{-k,n}$ is bounded and admits a convergent subsequence to α^{-k}. Taking such subsequence, we get $\max\{u(\alpha^{k,n}, \theta), u(\alpha^{-k,n}, \theta)\} \to u(\alpha^{-k}, \theta)$ for each θ. Using the dominated convergence theorem in a similar way as above, we get

$$v(\alpha^{k,n}, \alpha^{-k,n}) \to \int_\Theta u(\alpha^{-k}, \theta) dF \leq -\phi \operatorname{Var}[\theta_1] - \operatorname{Var}[\theta_2].$$

But the supremum $v^* > \phi \operatorname{Var}[\theta_1] + \operatorname{Var}[\theta_2]$, a contradiction.

Therefore, the sequence $(\alpha^{1,n}, \alpha^{2,n})$ is bounded and admits a convergent subsequence. By continuity, a maximum then exists.

To conclude the proof, note that the maximization problem (2) gives the Perfect Bayesian Equilibrium strategies of the common interest game where the sender chooses a strategy $\mu : \Theta \to \{1, 2\}$ and the receiver chooses $(\alpha^1, \alpha^2) \in \mathbb{R}^2 \times \mathbb{R}^2$ to maximize

$$\max_{\mu, \alpha} \int_\Theta \mathbb{1}[\mu(\theta) = 1] u(\alpha^1, \theta) + \mathbb{1}[\mu(\theta) = 2] u(\alpha^2, \theta) dF. \tag{3}$$

$\square$

Proposition 1 is a corollary of Lemma 2.

Proof of Proposition 1. Note first that $\alpha^1 = \alpha^2 = \mathbb{E}[\theta]$ is not a solution of the maximization problem (3), as any arbitrary partition of Θ and the best-reply to it would give strictly higher payoffs. This means that the solution to (3) is a non-constant μ. Moreover, the strategy $\mu : \Theta \to \{1, 2\}$ trivially satisfies the IVP. Therefore, a credible score exists. $\square$

B Proof of Remark 1 and Remark 2

Let s_1 denote the score that assigns a signal to $(0, 0)$ and $(0, 1)$ and another signal to $(1, 0)$ and $(1, 1)$. Let s_2 denote the score that assigns a signal to $(0, 0)$ and $(1, 0)$ and another signal to $(0, 1)$ and $(1, 1)$. It is immediate that the optimal score belongs to the set $\{s_1, s_2, s_D, s_d\}$. Let the payoffs associated with s_D, s_d, s_1 and s_2 be respectively, u_D, u_d, u_1 and u_2 so that:

$$\begin{aligned}
u_D &:= -2g(f(0,0), f(1,1));\\
u_d &:= -2g(f(1,0), f(0,1));\\
u_1 &:= -g(f(0,0), f(0,1)) - g(f(1,0), f(1,1));\\
u_2 &:= -g(f(0,0), f(1,0)) - g(f(0,1), f(1,1)),
\end{aligned}$$

where $g(x, y) := \frac{xy}{x+y}$.

Lemma 3. *If $f(0, 1) > f(0, 0)$, then score s_2 is not optimal.*

Proof. Suppose first that $f(1, 0) > f(1, 1)$. Simple algebra gives:

$$\begin{gathered}
u_2 < u_D \Leftrightarrow\\
g(f(0,0), f(1,0)) - g(f(0,0), f(1,1)) > g(f(0,0), f(1,1)) - g(f(0,1), f(1,1)).
\end{gathered}$$

As $g_x > 0$, then $f(1, 0) > f(1, 1)$ ensures that the left side of the last inequality is positive: at the same time, $f(0, 1) > f(0, 0)$ ensures that the right side is non-positive. We conclude that the last inequality holds and indeed $u_2 < u_D$. So for $f(1, 0) > f(1, 1)$, score s_2 is not optimal.

Suppose now that $f(1, 0) < f(1, 1)$. We proceed by contradiction. Suppose that $u_2 \geq$

$\max\{u_D, u_d\}$. Then

$$g(f(0,0), f(1,0)) + g(f(0,1), f(1,1)) \leq 2g(f(1,0), f(0,1)), \text{ and}$$
$$g(f(0,0), f(1,0)) + g(f(0,1), f(1,1)) \leq 2g(f(0,0), f(1,1)).$$

These 2 inequalities imply that the sum of the right sides must be larger than the sum of the left sides:

$$2g(f(0,0), f(1,0)) + 2g(f(0,1), f(1,1)) \leq 2g(f(1,0), f(0,1)) + 2g(f(0,0), f(1,1)) \Leftrightarrow$$
$$g(f(0,1), f(1,1)) - g(f(0,0), f(1,1)) \leq g(f(1,0), f(0,1)) - g(f(0,0), f(1,0)).$$

As $g_{xy}(\cdot) > 0$, then $f(0,1) > f(0,0)$ and $f(1,0) < f(1,1)$ together imply that the last inequality is violated. This contradiction implies that $u_2 < \max\{u_D, u_d\}$. Hence, for $f(1,0) < f(1,1)$, score s_2 is not optimal. As $f(1,0) \neq f(1,1)$ by assumption, the lemma follows. □

Proof of Remark 1. Lemma 3 establishes that if $f(0,1) > f(0,0)$, then score s_2 is not optimal. The same arguments can be used to show that also for $f(0,1) < f(0,0)$ score s_2 is not optimal. As $f(0,1) \neq f(0,0)$ by assumption, we conclude that score s_2 cannot be optimal. The proof that s_1 cannot be optimal follows the same steps and is omitted. We conclude that the optimal score is either s_D or s_d. The last part of the remark is immediate. □

Proof of Remark 2. Suppose parameters are such that s_d is optimal (the argument is identical if s_D is optimal). Consider a PBE such that $\mu(\theta) = s_d$. In such a PBE, $\mu(0,0) = 1$, $\mu(0,1) = \mu(1,0) = 2$ and $\mu(1,1) = 3$; $\alpha(1) = (0,0)$, $\alpha(2) = (\frac{f(1,0)}{f(1,0)+f(0,1)}, \frac{f(0,1)}{f(1,0)+f(0,1)})$ and $\alpha(3) = (1,1)$. Note that $u(\alpha(3),(1,0)) = u(\alpha(1),(1,0)) = -1$ hence

$$u(\alpha(2),(1,0)) \geq u(\alpha(1),(1,0)) \Leftrightarrow u(\alpha(2),(1,0)) \geq u(\alpha(3),(1,0)) \Leftrightarrow \frac{f(1,0)}{f(0,1)} \geq \sqrt{2} - 1,$$

while

$$u(\alpha(2),(0,1)) \geq u(\alpha(1),(0,1)) \Leftrightarrow u(\alpha(2),(0,1)) \geq u(\alpha(3),(0,1)) \Leftrightarrow \frac{f(1,0)}{f(0,1)} \leq \frac{1}{\sqrt{2}-1}.$$

A necessary condition for s_d to be credible is therefore that

$$\frac{f(1,0)}{f(0,1)} \in \left[\sqrt{2}-1, \frac{1}{\sqrt{2}-1}\right].$$

To conclude the proof it is sufficient to note that (a) this condition is also sufficient, as deviations for the sender are unprofitable upon observing some $\theta \in \{(0,0),(1,1)\}$ and (b)

$$\frac{f(1,0)}{f(0,1)} \in \left[\sqrt{2}-1, \frac{1}{\sqrt{2}-1}\right] \Leftrightarrow \frac{f(0,1)}{f(1,0)} \in \left[\sqrt{2}-1, \frac{1}{\sqrt{2}-1}\right]$$

□

C Proof of Proposition 2

For a score s, let $\alpha(m) = \mathbb{E}[\theta|m]$, let M be the image of s and $\alpha(M)$ the image of $\alpha(\cdot)$. Let $\Theta(a) = \{\theta : \alpha(s(\theta)) = a\}$. For any two points, $x, y \in \mathbb{R}^2$, with a slight abuse of notation, let $[x,y] = \text{conv}\,\{x,y\}$, $(x,y) = [x,y] \setminus \{x,y\}$ and $[x,y) = [x,y] \setminus \{y\}$. Finally, let $\ell(x,y)$ be the line connecting the points x, y.

The following lemma will be used throughout the proof.

Lemma 4. *Let $a, a' \in \mathbb{R}^2$. If $u(a,\theta) \geq u(a',\theta)$, then $u(a,\theta') > u(a',\theta')$ for all $\theta' \in [a,\theta)$.*

Proof. First assume that $a' \notin \ell(a,\theta)$. Take $\theta' \in [a,\theta)$. Note that

$$\begin{aligned} & -u(a,\theta) \leq -u(\theta,a') \\ \Rightarrow & \sqrt{-u(\theta,a)} \leq \sqrt{-u(\theta,a')} < \sqrt{-u(\theta,\theta')} + \sqrt{-u(\theta',a')}, \end{aligned} \tag{4}$$

where the last inequality holds by the triangle inequality and is strict because θ, θ' and a' are not collinear. Note also that

$$\begin{aligned}\sqrt{-u(\theta',\theta)}+\sqrt{-u(a,\theta')}=\sqrt{-u(a,\theta)}<\sqrt{-u(\theta,\theta')}+\sqrt{-u(\theta',a')}\\ \Rightarrow\ -u(a,\theta')<-u(a',\theta')\\ \Leftrightarrow\ u(a,\theta')>u(a',\theta'),\end{aligned}$$

where the equality holds as a, θ and θ' are collinear, and the first inequality follows from (4).

If instead $a' \in \ell(a,\theta')$, we must have $a' \notin (a,\theta]$, otherwise $u(a,\theta) < u(a',\theta)$. But then, either $a \in (a',\theta')$ or $\theta \in (\theta',a')$. In both cases, $u(a,\theta') > u(a',\theta')$. $\square$

We first consider the case in which all points in $\alpha(M)$ are isolated.

Lemma 5. *If all points in $\alpha(M)$ are isolated, then $s(\theta)$ is equivalent to a coarsely linear score.*

Proof. For any two $a, a' \in \alpha(M)$, let $\Theta^{\geq}(a,a') := \{\theta : u(a,\theta) \geq u(a',\theta)\}$. This set is a half-space:

$$u(\theta,a) \geq u(\theta,a') \Leftrightarrow -2\theta_1 a_1\phi + a_1^2\phi - 2\theta_2 a_2 + a_2^2 \geq -2\theta_1 a_1'\phi + a_1'^2\phi - 2\theta_2 a_2' + a_2'^2.$$

Similarly, let $\Theta^{=}(a,a') := \{\theta : u(a,\theta) = u(a',\theta)\}$. This set is a line.

If $|\alpha(M)| = 2$, the set $\Theta^{=}(a,a')$ determines the half-space defining a coarsely linear score.

Suppose there are three points $a^1, a^2, a^3 \in \alpha(M)$ and $m^i \in \alpha^{-1}(a^i)$ for $i = 1,2,3$ such that (i) $m^1 < m^2 < m^3$ and (ii) for any action $a' \in \alpha(M) \setminus \{a^1,a^2,a^3\}$, every $m \in \alpha^{-1}(a')$ satisfies $m > m^3$ or $m < m^1$.

Suppose that $\Theta^{=}(a^1,a^2)$ and $\Theta^{=}(a^2,a^3)$ are not parallel. Then $\Theta(a^2) \subseteq \Theta^{\geq}(a^2,a^1) \cap \Theta^{\geq}(a^2,a^3)$ and the set $\Theta^{\geq}(a^2,a^1) \cap \Theta^{\geq}(a^2,a^3)$ is a polyhedron with an extreme point at $\Theta^{=}(a^2,a^1) \cap \Theta^{=}(a^2,a^3)$.

Clearly $\{a^1,a^3\} \cap \Theta^{\geq}(a^2,a^1) \cap \Theta^{\geq}(a^2,a^3) = \emptyset$. Moreover, we can draw a curve from a^1 to a^3 in $\Theta \setminus (\Theta^{\geq}(a^2,a^1) \cap \Theta^{\geq}(a^2,a^3))$ consisting of straight vertical and horizontal lines. By the IVP, there must be θ' on that curve such that $s(\theta') = m^2$, a contradiction. $\square$

We consider next the case in which not all points in $\alpha(M)$ are isolated.

Lemma 6. *Let a be a limit point in $\alpha(M)$. Then int $\Theta(a) = \emptyset$.*

Proof. To establish that $\operatorname{int}\Theta(a) = \emptyset$, we proceed by contradiction. Suppose $\operatorname{int}\Theta(a) \neq \emptyset$ and let $\theta \in \operatorname{int}\Theta(a)$. Hence $u(a,\theta) \geq u(a',\theta)$ for all $a' \in \alpha(M)$. Let a'' be such that $u(a,\theta) = u(a'',\theta)$. Because $\theta \in \operatorname{int}\Theta(a)$, there is $\epsilon > 0$, such that for all $\theta' \in B_\epsilon(\theta)$, $\theta \in \Theta(a)$. Therefore, $(\theta, a''] \cap B_\epsilon(\theta)$ is not empty. But by Lemma 4, $\theta' \in (\theta, a'']$ implies $u(a'',\theta') > u(a,\theta')$, contradicting $\theta' \in \Theta(a)$. Hence $u(a,\theta) > u(a',\theta)$ for all $a' \in \alpha(M) \setminus \{a\}$.

Now we argue that $\operatorname{int}\Theta(a)$ is convex. Let $\theta, \theta' \in \operatorname{int}\Theta(a)$ and $\theta'' \in [\theta, \theta']$. First observe that

$$u(\theta, a) > u(\theta, a') \Leftrightarrow -2\theta_1 a_1 \phi + a_1^2 \phi - 2\theta_2 a_2 + a_2^2 > -2\theta_1 a_1' \phi + a_1'^2 \phi - 2\theta_2 a_2' + a_2'^2. \quad (5)$$

The inequality is preserved under convex combinations, so $u(a,\theta'') > u(a',\theta'')$ for all $a' \in \alpha(M) \setminus \{a\}$, and thus $\theta'' \in \Theta(a)$.

We show next that $\theta'' \in \operatorname{int}\Theta(a)$. Take $\epsilon > 0$, such that $B_\epsilon(\theta) \subset \operatorname{int}\Theta(a)$. If $\theta'' \in B_\epsilon(\theta)$, we are done. Suppose $\theta'' \notin B_\epsilon(\theta)$. Take two points $\theta^1, \theta^2 \in B_\epsilon(\theta)$ such that $\theta'' \notin [\theta^i, \theta']$ for $i = 1, 2$, and $\theta \in (\theta^1, \theta^2)$. This implies that θ^1, θ^2 and θ' are not collinear.[11] In that case, the convex hull $\operatorname{conv}\{\theta^1, \theta^2, \theta'\} \subseteq \Theta(a)$ has a non-empty interior and contains θ''. Since θ'' is not on the boundary of $\operatorname{conv}\{\theta^1, \theta^2, \theta'\}$, it is in its interior. There exists thus an $\eta > 0$ such that $B_\eta(\theta'') \subseteq \operatorname{conv}\{\theta^1, \theta^2, \theta'\} \subseteq \Theta(a)$. Therefore, $\theta'' \in \operatorname{int}\Theta(a)$, and $\operatorname{int}\Theta(a)$ is convex.

If $\operatorname{int}\Theta(a)$ is not empty and convex, then the boundary of $\Theta(a)$ has measure zero in $\mathbb{R}^2$ (see e.g., Lang, 1986). Moreover, since $\mathbb{E}[\theta | s(\theta) = m] = a$ for all $m \in \{m' \in M : \alpha(m') = a\}$, we have

$$\mathbb{E}[\theta | \theta \in \Theta(a)] = a.$$

Therefore,

$$\mathbb{E}[\theta | \theta \in \Theta(a)] = \mathbb{E}[\theta | \theta \in \operatorname{int}\Theta(a)] = a,$$

which implies $a \in \operatorname{int}\Theta(a)$. But then, because a is a limit point of $\alpha(M)$, it means that $\operatorname{int}\Theta(a)$ intersects with $\alpha(M)$ at a point different than a, i.e., there is a point $a' \in \alpha(M)$

[11] For example, two points whose segment $[\theta^1, \theta^2] \subseteq B_\epsilon(\theta)$ is perpendicular to $[\theta, \theta']$ satisfy these conditions.

and associated message m' with $\alpha(m') = a'$ such that $0 > u(a', a) \geq u(a', \alpha(m')) = 0$. A contradiction. Hence, $\operatorname{int}\Theta(a) = \emptyset$. $\square$

Lemma 7. *Let a be a limit point in $\alpha(M \setminus \{\inf M, \sup M\})$. Then $\Theta(a) = \ell(\theta, \theta')$ for some θ and θ'. Moreover, for all limit points a in $\alpha(M \setminus \{\inf M, \sup M\})$, the lines $\Theta(a)$ are parallel.*

Proof. First, we show that there are θ and θ' such that $\Theta(a) \subseteq \ell(\theta, \theta')$.

From the proof of Lemma 1, $|\Theta(a)| > 1$ and therefore $\Theta(a) \neq \{a\}$.

Note that a cannot be an extreme point of $\operatorname{conv}\Theta(a)$ as $\mathbb{E}[\theta|\theta \in \Theta(a)] = a$ and $\Theta(a) \neq \{a\}$. This means that there exist $\theta, \theta' \in \Theta(a)$ such that $a \in [\theta, \theta']$.

By Lemma 4, we can assume that for $\theta^\dagger \in \{\theta, \theta'\}$ we have $u(\theta^\dagger, a) > u(\theta^\dagger, a')$ for all $a' \in \alpha(M) \setminus \{a\}$. Otherwise, we can just take a smaller interval contained in $[\theta, \theta']$.

Suppose there is $\theta'' \notin \ell(\theta, \theta')$ such that $\theta'' \in \Theta(a)$. Again, we can take θ'' such that $u(\theta'', a) > u(\theta'', a')$ for all $a' \in \alpha(M) \setminus \{a\}$. As argued in the proof of Lemma 6, the set $\operatorname{conv}\{\theta, \theta', \theta''\} \subseteq \Theta(a)$. Since these points are not aligned, $\operatorname{conv}\{\theta, \theta', \theta''\}$ has a non-empty interior and therefore $\operatorname{int}\Theta(a)$ has a non-empty interior. A contradiction.

To prove that $\Theta(a) = \ell(\theta, \theta')$, it is then enough to show that the set $\Theta(a)$ is unbounded in both directions. To see this, take some $\theta \in \Theta(a)$ and let $m = s(\theta)$. We can repeat the same argument as in Lemma 1. Let m_1 and m_2 satisfy $m_1 < m < m_2$ and pick θ^1 and θ^2 such that $s(\theta^1) = m_1$ and $s(\theta^2) = m_2$.

If $\Theta(a)$ is bounded in one direction, we can find a curve consisting of straight horizontal and vertical lines such that this curve does not intersect with $\Theta(a)$. By the IVP, there must be θ' on that curve such that $s(\theta') = m$ and therefore $\theta' \in \Theta(a)$, a contradiction. Therefore, $\Theta(a) = \ell(\theta, \theta')$.

Let a and a' be limit points of $\alpha(M \setminus \{\inf M, \sup M\})$ such that $a \neq a'$. Because $\Theta(a) \cap \Theta(a') = \emptyset$, the lines $\Theta(a)$ and $\Theta(a')$ must be parallel. $\square$

Let A_I be the set of isolated points in $\alpha(M)$ and A_L be the set of limit points in $\alpha(M)$. Denote by $\ell_s(a)$ the line that goes through a and has the same slope as $\Theta(a')$ for some $a' \in A_L$.

Lemma 8. *If there are some limit points in $\alpha(M)$, then all points in $\alpha(M)$ are limit points.*

Proof. Let $\Theta^\dagger = \cup_{a \in \operatorname{cl} A_L} \Theta(a) = \cup_{a \in \operatorname{cl} A_L} \ell_s(a)$.

Take $a \in \arg\max_{a' \in A_I} \sup_{\theta \in \Theta^\dagger} u(a', \theta)$ and $\theta^\dagger \in \arg\max_{\theta \in \Theta^\dagger} u(a, \theta)$. The points a and $\theta^\dagger$ are the two points in A_I and $\Theta^\dagger$ with minimal (weighted) distance between the two. Moreover, this distance is bounded away from zero either by the definition of isolated points if $\theta^\dagger \in A_L$ or by the optimality of generating an action in A_L for states arbitrarily close to $\theta^\dagger$ if $\theta^\dagger \notin A_L$.

Note that $\theta^\dagger$ is on the boundary of $\Theta^\dagger$, otherwise there is another point in $\Theta^\dagger$ closer to a. Take $\tilde{a} \in \mathrm{cl} A_L$ such that $\theta^\dagger \in \ell_s(\tilde{a})$. Because the $\Theta^\dagger$ is a union of lines, if $\theta^\dagger \in \ell_s(\tilde{a})$ is on the boundary of $\Theta^\dagger$, then $\ell_s(\tilde{a})$ is on the boundary of $\Theta^\dagger$. We can therefore find a sequence $\theta^n \notin \Theta^\dagger$ with $\theta^n \to \tilde{a}$. By definition of isolated points, there is $\epsilon > 0$ such that $u(a, \tilde{a}) < -\epsilon$ for all $a \in A_I$. But then for n large enough, θ^n prefers to induce an action in A_L, a contradiction. □

Lemma 9. *If there are some limit points in $\alpha(M)$, any credible score is equivalent to a linear score.*

Proof. By Lemma 8, if there are some limit points in $\alpha(M)$, then all points in $\alpha(M)$ are limit points.

If $\inf M \notin M$ and $\sup M \notin M$, then by Lemma 7, the score is equivalent to a credible score.

To conclude the proof, we will show that $\inf M \notin M$ and $\sup M \notin M$. Suppose it is not the case and that $m = \min M$ exists. By Lemma 8, because there are some limit points in $\alpha(M)$, $\alpha(m)$ is a limit point of $\alpha(M)$. Therefore, there is a neighborhood of $\alpha(m)$, denote it $\Theta^\dagger$, such that for all $\theta \in \Theta^\dagger$, it is the case that $\sup_{a \in A_L} u(\theta, a) > \sup_{a \in A_I} u(\theta, a)$ and for all $a \in \Theta^\dagger \cap \alpha(M)$, it is the case that $a \in A_L$. That is, types in $\Theta^\dagger$ are closer to points in A_L than to points in A_I.

Take a point in $\theta \in \ell_s(\alpha(m)) \cap \Theta^\dagger$. It cannot be that $\alpha(s(\theta)) \in A_I$ by definition of $\Theta^\dagger$. It also cannot be that $\alpha(s(\theta)) \in A_L \setminus \{\alpha(m)\}$ as $\theta \in \ell_s(\alpha(m))$. Therefore, $\alpha(s(\theta)) = \alpha(m)$ and there is more than one point in $\Theta(\alpha(m))$. By a similar argument as above, it must be that $\Theta(\alpha(m)) \subseteq \ell_s(\alpha(m))$.

Let Θ^+ and Θ^- denote the two open half-spaces defined by the line $\ell_s(\alpha(m))$. Suppose $a^+ \in \Theta^+$ and $a^- \in \Theta^-$ such that $a^+, a^- \in \Theta^\dagger \cap \alpha(M)$, i.e., there are actions played in equilibrium in A_L that are on both sides of $\ell_s(\alpha(m))$. Note that $\ell_s(a^-) \subset \Theta^-$.

Suppose without loss of generality that $m^+ = s(a^+) > m^- = s(a^-)$. By definition, $m^- > m$. Take two points $\theta^+ \in \ell_s(a^+)$, $\theta^m \in \Theta(\alpha(m))$ such that $\theta^+ > \theta^m$ or $\theta^+ < \theta^m$. We can draw a curve between θ^+ and θ^m that is entirely in Θ^+ (except at θ^m) that consists only of straight horizontal and vertical lines. By IVP, there must be θ' on that curve such that $s(\theta') = m^-$. But $\theta' \in \Theta^+$ and $\notin \ell_s(a^-) = \Theta(a^-)$, a contradiction.

Therefore all $\theta \in \Theta^\dagger \cap \alpha(M)$ are in the same half-space, say Θ^-. But types in $\Theta^+ \cap \Theta^\dagger$ should prefer sending messages that induce $a \in A_L$, contradicting that $\Theta(a) \subseteq \ell_s(a)$. □

Proof of Proposition 2. Proposition 2 follows from Lemmas 5 and 9. □

D Proof of Proposition 3

Proof of Proposition 3. For any strategy $s(\theta) = \beta'\theta$, we have the unconditional distribution over messages m induced by the score s, $m \sim N(0, \sigma_s^2)$ where $\sigma_s^2 = \beta_1^2\sigma_1^2 + \beta_2^2\sigma_2^2 + 2\beta_1\beta_2\sigma_{12} = \beta'\Sigma\beta$. We also have that $Cov(\theta_i, m) = \sigma_{is} = \beta_i\sigma_i^2 + \beta_j\sigma_{12}$. Therefore, $(\sigma_{1s}, \sigma_{2s})' = \Sigma\beta$.

The payoff of the sender can be rewritten, up to a constant, as

$$-a'\Phi a + 2a'\Phi\theta.$$

Therefore, the ex-ante payoff – given that the best-reply to m is $\alpha(m) = \frac{\Sigma\beta}{\beta'\Sigma\beta}m$ – is

$$\begin{aligned}
&-\mathbb{E}[\alpha(m)'\Phi\alpha(m) + 2\alpha(m)'\Phi\theta] \\
&= -\mathbb{E}[\frac{\beta'\Sigma}{\beta'\Sigma\beta}m\Phi\frac{\Sigma\beta}{\beta'\Sigma\beta}m - 2\frac{\beta'\Sigma}{\beta'\Sigma\beta}m\Phi\theta] \\
&= -\frac{\beta'\Sigma\Phi\Sigma\beta}{\beta'\Sigma\beta},
\end{aligned}$$

where the last equality follows from $\mathbb{E}[m^2] = \beta'\Sigma\beta$ and $\mathbb{E}[\theta m] = \Sigma\beta$. The matrix $\Sigma\Phi\Sigma$ is positive semidefinite and symmetric. Therefore, $\frac{\beta'\Sigma\Phi\Sigma\beta}{\beta'\Sigma\beta}$ is a generalized Rayleigh quotient (see e.g., Parlett, 1998, Chapter 15) and the two stationary points of $\frac{\beta'\Sigma\Phi\Sigma\beta}{\beta'\Sigma\beta}$ are the eigenvectors of $\Sigma^{-1}(\Sigma\Phi\Sigma) = \Phi\Sigma$, i.e., the points β such that there is $\lambda \in \mathbb{R}$ such that $\Phi\Sigma\beta = \lambda\beta$. Moreover, as generalized Rayleigh quotients attain a maximum and a minimum, one of the stationary points must correspond to a maximizer, the other to a minimizer.[12]

[12] If the state had more than two dimensions, there would be more stationary points/eigenvectors; yet, it would still be the case that one of the eigenvectors corresponds to a maximizer of the Rayleigh quotient, another

The equilibrium problem can be expressed as follows. Given a belief that the sender uses a linear strategy β, the receiver chooses $\alpha(m) = \frac{\Sigma\beta}{\beta'\Sigma\beta}m$. In equilibrium, the sender chooses a signal m for each realization of θ:

$$\max_m -\frac{\beta'\Sigma m\Phi\Sigma\beta m}{(\beta'\Sigma\beta)^2} + 2\frac{\beta'\Sigma m\Phi\theta}{\beta'\Sigma\beta}.$$

The objective function is quadratic in m and therefore the maximizer must satisfy the first order condition:

$$m = \beta'\Sigma\beta\frac{\beta'\Sigma\Phi}{\beta'\Sigma\Phi\Sigma\beta}\theta.$$

Therefore, any equilibrium strategy must satisfy

$$\beta' = \beta'\Sigma\beta\frac{\beta'\Sigma\Phi}{\beta'\Sigma\Phi\Sigma\beta} \Leftrightarrow \beta = \frac{\beta'\Sigma\beta}{\beta'\Sigma\Phi\Sigma\beta}\Phi\Sigma\beta.$$

Take any equilibrium strategy β. From the equilibrium condition, β is an eigenvector of $\Phi\Sigma$ with eigenvalue $\frac{\beta'\Sigma\Phi\Sigma\beta}{\beta'\Sigma\beta}$.

Conversely, take an eigenvector β of $\Phi\Sigma$, with eigenvalue λ. Plugging in the equilibrium condition, we get

$$\beta = \beta'\Sigma\beta\frac{\Phi\Sigma\beta}{\beta'\Sigma\Phi\Sigma\beta} \Leftrightarrow \beta = \frac{\beta'\Sigma\beta}{\lambda\beta'\Sigma\beta}\lambda\beta, \tag{6}$$

where the equivalence follows from $\Phi\Sigma\beta = \lambda\beta$ and $\beta'\Sigma\Phi = \lambda\beta'$. Equation (6) is satisfied and therefore β is an equilibrium strategy. $\square$

to the minimizer.