Managing the expectations of buyers with reference-dependent preferences

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Abstract

I consider a model of monopoly pricing where a risk-neutral firm makes an offer to a buyer with reference-dependent preferences. The reference point is the ex-ante probability of trade and the buyer exhibits an attachment effect: the higher his expectations to buy, the higher his willingness-to-pay. When the buyer's valuation is private information, a unique equilibrium exists where the firm plays a mixed strategy and its profits are the same as in the reference-independent benchmark. The equilibrium always entails inefficiencies: even as the firm's information converges to complete information, it mixes on a non-vanishing support and the probability of no trade is greater than zero. Finally, I show that when the firm can obtain costless signals on the buyer's valuation, it can do strictly better than in the reference-independent benchmark by leveraging the uncertainty generated by a noisy learning strategy. However, this advantage vanishes as the attachment effect grows large.

The purpose of this paper is to study a monopoly pricing model where the buyer exhibits a specific type of reference-dependent preferences. I assume that he values a good more when

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he expects to own it. For instance, a buyer can get emotionally attached to a good he expects to buy, e.g., a house, a car or clothes, and thus finds it harder to refuse an offer and walk away. A job applicant, expecting to be employed, can build expectations about the prospects of a different lifestyle or social status, which can reduce his willingness to refuse an offer. More generally, this attachment effect is an "expectation-based endowment effect" and is a prediction of expectation-based reference-dependent preferences (Kőszegi and Rabin, 2006).

When can a firm benefit from facing a buyer with an attachment effect and what is its consequence on a firm's pricing strategy? To study this question, I adapt the monopoly model of Heidhues and Kőszegi (2014). A firm sells an indivisible good to a buyer by making a take-it-or-leave-it offer. To capture the attachment effect, the buyer's willingness-to-pay (WTP) increases linearly in the ex-ante probability of trading. Following the literature on expectation-based reference-dependent preferences, the buyer plays a Preferred Personal Equilibrium: when setting expectations, he correctly anticipates the firm's strategy and his own action and selects the most favourable plan of action.[1] This model has two main features. First, the demand is endogenous: the probability of buying and hence the demand depends on the firm's strategy. Second, the firm has a commitment problem. There is a tension between offering low prices to induce expectations to buy and high prices to take advantage of a higher willingness-to-pay.

I consider three information environments: (1) the valuation is the buyer's private information, (2) the buyer's valuation is common knowledge, and (3) the firm can learn about the valuation.

Proposition 2 characterises the unique equilibrium of the game when valuations are private information. In equilibrium, the firm chooses the mixed strategy such that the resulting demand is unit-elastic on the support. That is, it creates the demand that makes it indifferent between any price on the support. This preserves the incentives for the mixed strategy and solves the firm's commitment problem. In this case, the firm does not benefit from the attachment effect and equilibrium profits are independent of its strength. Indeed, types below the support of the mixed strategy only face prices above their valuation in equilibrium. By the PPE requirement, they cannot expect to buy and behave as if they have no attachment effect. This implies that the profits from the lowest price on the support must be the same as in the reference-independent benchmark[2], pinning down equilibrium profits. The commitment

[1] See for example Heidhues and Kőszegi (2014), Kőszegi and Rabin (2009), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018) for papers using this selection.

[2] Throughout the paper, the reference-independent model refers to an equivalent model with no attachment effect, i.e., the WTP is the same as the valuation.

problem created by the attachment effect has thus two effects here: the firm must use random prices to overcome it and it does not benefit from having reference-dependent buyers.

In section 2.1, I characterise the firm's pricing strategy when it knows the buyer's valuation in two different ways, and get contrasting results. First, I use the incomplete information characterisation to study convergence to complete information. As the distribution over types concentrates on a singleton, the equilibrium strategy converges to a mixed strategy on a non-vanishing support. Moreover, the probability of no trade is bounded away from zero. This follows from the incomplete information characterisation. In order to create a unit-elastic demand, the firm induces a large variation in the trading probability of almost identical types. This results in a positive probability of no trade. This problem is more severe for a stronger attachment effect: the probability of trading converges to zero as the attachment effect grows large.

However, there is a discontinuity at the limit: when the valuation is common knowledge, no equilibrium exists. Indeed, it is impossible for the firm to overcome its commitment problem. Any pricing strategy where the buyer is willing to accept increases his WTP above the price played in equilibrium.[3]

In Heidhues and Kőszegi (2014), the firm can commit to a price distribution and the valuation is common knowledge. The optimal strategy is to randomise over prices. The low prices in the support ensure that the buyer expects to buy with positive probability. This increases the WTP through the attachment effect. The higher prices in the support exploit this higher WTP to increase profits. In their model, the firm is strictly better off facing a buyer with an attachment effect and the equilibrium is efficient in the sense that there is trade with probability one. However, this strategy cannot hold without commitment because the firm has no incentives to offer the low prices in the support when higher prices are also accepted.

In contrast, I consider a firm that cannot commit to a price distribution and the buyer's valuation is private information. I show that inefficiencies are a general feature of this model and that the firm does not benefit from facing a buyer with an attachment effect. Moreover, an equilibrium does not exist when the valuation is common knowledge. This shows that the commitment assumption is key to both existence and benefiting from the attachment effect. On the other hand, and through a different channel, the model without commitment does predict random prices like in Heidhues and Kőszegi (2014). It also turns out that the strategy in the incomplete information limit is the same as the strategy under complete information and

[3] Existence issues with PPE in strategic settings were already pointed out by Dato et al. (2017).

commitment, albeit with different limit profits.

This first set of results shows that when the firm cannot commit to a pricing strategy, it cannot use the buyer's reference dependence to increase its profits. In section 3, I allow the firm to optimally learn about the valuation. For example, an internet platform gathers information about a consumer, a salesman asks some questions to a buyer or a hiring committee designs a selection process. In this new environment, the firm first designs a publicly observed information structure, privately observes a signal realisation then makes an offer. In its optimal learning strategy, the firm chooses a noisy test and is strictly better off than in the reference-independent benchmark. Intuitively, when learning is noisy, the firm can create random price offers like in Heidhues and Kőszegi (2014). At the same time, the firm is uncertain about the type it is facing unlike in Heidhues and Kőszegi (2014) so offering low prices can be credible.

In the optimal information structure, each buyer generates signals whose associated posterior beliefs are either higher or lower than his type. Therefore, each buyer receives low price offers, inducing expectations to buy, and high price offers, taking advantage of the higher WTP. However, there is a limit to this logic because low types can be pooled with fewer lower types. They thus expect higher prices and must expect to trade less often. The optimal information structure thus creates downward distortions: lower types trade with a lower probability. From a technical point of view, I solve an information design in the spirit of Bayes Correlated Equilibrium (Bergemann and Morris, 2016): the firm maximises over conditional price distributions subject to obedience constraints with the additional constraint that the WTPs are determined endogenously. The problem without obedience constraints correspond to the one in Heidhues and Kőszegi (2014).

I also find that increasing the strength of the attachment effect has a non-monotonic effect on profits. With a stronger attachment effect, the firm can increase the buyer's WTP more. However, it also increases the firm's commitment problem by making it more tempting to increase prices. To maintain the credibility of the price distribution, the firm must create more inefficiencies. When the attachment effect is arbitrarily large, the profits converge to the reference-independent profits.

Relation to the literature

This paper is part of a large literature that studies the implication of rational expectations as the reference point in reference-dependent preferences, following Kőszegi and Rabin (2006). The two closest papers in this literature are Heidhues and Kőszegi (2014) and Eliaz and

Spiegler (2015).

The model with complete information is similar to Heidhues and Kőszegi (2014). They show that if the valuation is known, the firm benefits from creating a distribution over prices. The modelling difference is that I do not allow the firm to commit to a price distribution. I also show that an imperfect learning strategy provides a foundation for the stochastic pricing strategy without commitment. Eliaz and Spiegler (2015) look at a more abstract model that nests the complete information environment with commitment of this paper as a special case. They show that uniqueness of the PE can be guaranteed through a first-order stochastic dominance property that is useful in this paper. Rosato (2016) also studies a monopoly pricing model where the uncertainty is used to exploit expectation-based reference-dependent preferences. There, the monopolist commits to the limited availability of substitutes to induce the expectations of buying.

The last section is related to the literature on optimal learning and price discrimination. Bergemann et al. (2015) characterise all the combination of consumers' surplus and monopoly profit after some learning of the firm. I depart from their framework by introducing reference-dependent preferences. Where in their model the optimal learning strategy is to perfectly learn the valuation, introducing reference-dependent preferences incentivises the firm to create a stochastic environment.

Roesler and Szentes (2017) and Condorelli and Szentes (2020) look at environments where an agent designs an optimal learning strategy taking into account the effect of information acquisition on the other agent's strategy. Here, the firm designs its optimal learning strategy taking into account its effect on the buyer's preferences. Finally, there are links to the literature on optimal disclosure with a behavioural audiences as it is concerned with the design of the information environment with non-standard preferences, see e.g., Lipnowski and Mathevet (2018); Lipnowski et al. (2020); Levy et al. (2020).

1 The model

There is one firm and one buyer. The firm makes makes a take-it-or-leave-it offer $p \in \mathbb{R}$ for an indivisble good that the buyer can either accept, $a = 1$, or reject, $a = 0$. The buyer's payoff is

$$u(p, v, a|r) = a(v - p) - \lambda \cdot \mathbb{1}[a < r]$$

where $p \in \mathbb{R}$ is the price offered, v is an exogenous valuation, $a \in \{0, 1\}$ is the acceptance decision and $r \in \{0, 1\}$ where $r = 1$ stands for "expecting to accept", and $r = 0$ for "expecting to reject". Here, the buyer "pays" a penalty λ whenever he rejects an offer he was expecting to accept. Like in Kőszegi and Rabin (2006), I allow the reference point to be stochastic. The reference point is then $q \in [0, 1]$ which stands for the probability of accepting. The utility of buyer v is written as

$$u(p, v, a|q) = q \cdot \big(a(v - p) - \lambda \mathbb{1}[a < 1]\big) + (1 - q) \cdot \big(a(v - p) - \lambda \mathbb{1}[a < 0]\big)$$

The firm's payoff is

$$\pi(p, a) = a \cdot p$$

The buyer knows v. The firm only knows that $v \sim G$, where G denotes a cdf. It admits a strictly positive density g on the support $V = [\underline{v}, \overline{v}]$, $\underline{v} \geq 0$. I use γ to denote the probability measure associated with G: for any measurable set A, $Pr[v \in A] = \gamma(A)$. I will often refer to a valuation v as the buyer's type. The assumption that the firm has no cost is a normalisation given that $v \geq 0$, i.e., there is a positive surplus with any type.

Buyer's behaviour Given his valuation v and his reference point q, the buyer's payoffs from accepting and refusing at price p are

$$u(p, v, a = 1|q) = v - p$$
$$u(p, v, a = 0|q) = 0 - \lambda \cdot q$$

Therefore, he optimally plays a cutoff strategy: he accepts an offer p if and only if $p \leq v + \lambda q$.[4] I denote the buyer's optimal strategy $a^*(p, v|q) = \mathbb{1}[p \leq v + \lambda q]$.

Following Kőszegi and Rabin (2006), the buyer forms his reference point based on the correct expectations of trading. I assume that the buyer first learns his type, then forms his expectations based on some price distribution F. The reference point is thus formed after learning his own type but before the price realisation. Therefore, different types can have different expectations of trading. A Personal Equilibrium (PE) is a reference point q such that the probability of trading is consistent with the optimal strategy given the reference point.

[4]Here, I assume that, when indifferent, the buyer accepts the price offer. Allowing for mixed strategy would not change any of the results.

Definition 1. *Given a price distribution F, $(Q_v)_v$ is a profile of Personal Equilibria if for each $v \in V$, Q_v satisfies*

$$Q_v = \int_{\mathbb{R}} a^*(p, v|Q_v) dF(p)$$

and $a^*(p, v|Q_v) = \mathbb{1}[p \leq v + \lambda Q_v] \in \arg\max u(p, v, a|Q_v)$.

In a PE, the buyer with valuation v correctly anticipates how his expectations change his strategy and how his strategy changes his expectations. The PE Q_v depends on the type but also on the distribution over prices. Therefore, the buyer's behaviour will depend directly on the firm's strategy.

The expected utility of type v, for a given PE Q_v and price distribution $F(p)$ is

$$W(v|F, Q_v) = \int_{-\infty}^{v+\lambda Q_v} (v - p) dF(p) + \int_{v+\lambda Q_v}^{+\infty} -\lambda Q_v \, dF(p)$$

For any prices in $(-\infty, v + \lambda Q_v]$, the buyer accepts the offer and gets a utility $v - p$. For prices larger than $v + \lambda Q_v$, the buyer rejects the offer and gets a loss of $-\lambda Q_v$.

Because there can be multiple PEs, I assume the buyer plays his Preferred Personal Equilibrium (PPE). The PPE is the Personal Equilibrium that gives the highest expected utility (Kőszegi and Rabin, 2006, 2007).

Definition 2. *Given a price distribution F, $(Q^*_v)_v$ is a profile of Preferred Personal Equilibria if for each $v \in V$, $Q^*_v \in \arg\max_{Q_v \in PE} W(v|F, Q_v)$.*

This (Personal) equilibrium selection is common in the literature using Personal Equilibria.[5] Its motivation is based on an introspection interpretation of the PE. The buyer can entertain multiple expectations of trading but cannot fool himself: his reference point must be correct given his optimal behaviour. Then, if he can "choose" amongst multiple reference points, he would choose the one with the highest expected utility.

It will be useful to think of the PE or PPE as the cutoff price it generates.

Definition 3. *Given F a distribution over prices and PE Q_v, the PE cutoff price of type $v \in V$ is $\hat{p}(v) = v + \lambda Q_v$. Given PPE Q^*_v, the PPE cutoff price is $p^*(v) = v + \lambda Q^*_v$.*

The PPE cutoff price determines buyer v's willingness-to-pay (WTP). Note also that we have $Q^*_v = F(p^*(v))$, as buyer v accepts any price below $p^*(v)$. In the rest of the paper, the valuation refers to a buyer's type v and his willingness-to-pay to his PPE cutoff price, $p^*(v)$.

[5] See e.g., Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018).

Given a profile of PPE $(Q^*_v)_v$, let $V^*(p) = \{v \in V : p \leq p^*(v)\}$ be the set of types accepting price p. Define $v^*(p) = \inf\{v : v \in V^*(p)\}$, the lowest type in $V^*(p)$.

Equilibrium The firm's expected profits given the profile of PPE $(Q^*_v)_v$ are $\mathbb{E}[\pi(p)|(Q^*_v)_v] = p\,\gamma(V^*(p))$. I can now define an equilibrium in this model.

Definition 4. *A profile of strategy and reference points $\big(F(p), (Q^*_v)_v\big)$ is an equilibrium if for each $v \in V$, Q^*_v is type v's PPE given F and for each $p \in \operatorname{supp} F$, $p \in \arg\max_{\tilde{p}} \mathbb{E}[\pi(\tilde{p})|(Q^*_v)_v]$.*

In equilibrium, each buyer v forms his expectations based on on the firm's equilibrium strategy and his type and the firm's strategy is a best response to the buyers' PPEs.

Comment: Utility function The utility function I use allows me to capture an attachment effect in the simplest possible way. With this utility function, the agent pays a penalty λ weighted by the probability of accepting q when he does not accept the offer. The original specification of Kőszegi and Rabin (2006) allows for a reference point that depends both on the distribution over consumption and price paid. Here, the utility function is similar to ones used in the literature with loss-aversion in one dimension only.[6] Having loss-aversion in one dimension only allows to cleanly isolate an effect of the reference-dependent preferences, for example aversion to price increases or in this case, the attachment effect. The main difference with the rest of the literature is that the loss depends only on the probability of trading and not on the valuation.

1.1 Characterisation of the PPE

Proposition 1 establishes two properties of the PPE. First, the PPE cutoff is the smallest of the PE cutoffs. Second, it can be characterised with a sort first-order stochastic dominance condition. If $p^*(v)$ is the PPE cutoff then $F(p)$ must be strictly "first-order stochastically dominated" by the distribution $\frac{p-v}{\lambda}$ on $(-\infty, p^*(v)]$. The proof of Proposition 1 also establishes existence of the PPE.

Proposition 1. *For a fixed type v and distribution F, these three statements are equivalent:*

[6] For example, section 4.1 in Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Carbajal and Ely (2016), Rosato (2020) or Spiegler (2012).

- *$p^*(v)$ is PPE cutoff price*
- $p^*(v) = \min\{p : F(p) = \frac{p-v}{\lambda}\}$
- $v - p^*(v) = -\lambda F\big(p^*(v)\big)$ *and for all* $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda}$

Proof. Fix a type v. First, I show that the PPE cutoff is the lowest PE cutoff. Fix two PE, Q_1, Q_2 and their respective PE cutoffs, p_1, p_2. Then,

$$\begin{aligned} v - p_1 &= -\lambda F(p_1) \\ \text{and } v - p_2 &= -\lambda F(p_2) \end{aligned}$$

Note that we have $F(p_1) \neq F(p_2)$, for otherwise $p_1 = p_2$. The expected utility at PE Q_i is

$$W(v|Q_i) = \int_{-\infty}^{p_i} (v-p)dF(p) + \int_{p_i}^{+\infty} -\lambda F(p_i)dF(p)$$

Using the equality defining the cutoff, p_1 is preferred to p_2 if and only if

$$\begin{aligned} \int_{-\infty}^{p_1} (v-p)dF(p) + (1-F(p_1))(v-p_1) &\geq \int_{-\infty}^{p_2} (v-p)dF(p) + (1-F(p_2))(v-p_2) \\ \Leftrightarrow \int_{p_1}^{p_2} p\, dF(p) &\geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2F(p_2) - p_1F(p_1) - \int_{p_1}^{p_2} F(p)dp &\geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2 - p_1 &\geq \int_{p_1}^{p_2} F(p)dp \end{aligned}$$

Integrating by part on the third line. Because $F(p_1) \neq F(p_2)$, this is satisfied if and only if $p_1 < p_2$.

Therefore, $p^*(v) = \min\{p : v - p = -\lambda F(p)\}$. By applying Tarski's fixed point theorem on the non-decreasing function $p \to v + \lambda F(p)$ on the domain $[v, v+\lambda]$, the $\min$ over fixed points is well-defined.

We can now show $p^*(v) = \min\{p : v - p = -\lambda F(p)\} \Leftrightarrow F(p) > \frac{p-v}{\lambda}$ for all $p < p^*(v)$ and $F(p^*(v)) = \frac{p^*(v)-v}{\lambda}$.

($\Rightarrow$) If $p^*(v)$ is a PPE and we have some $\hat{p} < p^*(v)$ with $F(\hat{p}) \leq \frac{\hat{p}-v}{\lambda}$. We can then apply Tarski's fixed point theorem on the non-decreasing function $p \to v + \lambda F(p)$ on the domain $[v, \hat{p}]$. We would then find another PE smaller than $p^*(v)$, a contradiction.

($\Leftarrow$) If for all $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda}$, then there are no other PE smaller $p^*(v)$. $\square$

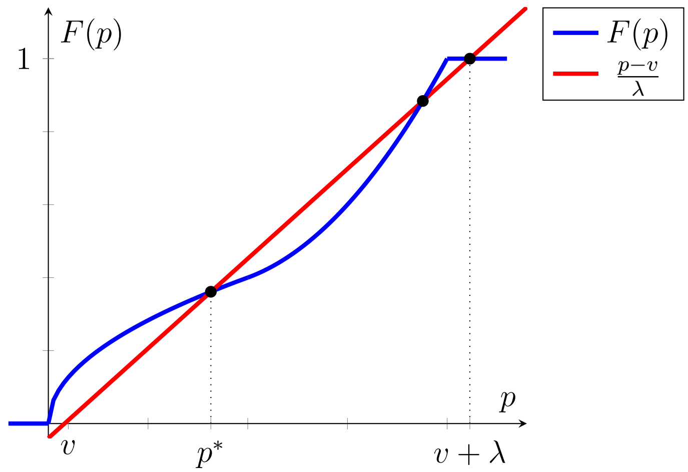

Figure 1: Each intersection of the blue and red curve is a PE. The lowest intersection, p^*, is the PPE.

The PPE cutoff price is the smallest of the PE cutoff prices because for any distribution, this cutoff is weakly above the valuation v. Therefore, the lowest PE cutoff minimises trade when $p > v$, i.e., when the buyer has a negative utility. This also implies the first-order stochastic dominance property. It simply guarantees that no other PEs exist below the PPE. This kind of condition was introduced by Eliaz and Spiegler (2015). The tractability and economic interpretation of this condition will be useful in section 3 when we will design the firm's optimal learning strategy.

2 Incomplete information

In this section, I first characterise the equilibrium when the valuation is private information. In section 2.1, I study the equilibrium when the set of types converges to a singleton and when the valuation is common knowledge.

To simplify the analysis in this section, I restrict attention to strictly concave reference-independent profits.

Assumption 1. *The function* $p(1 - G(p))$ *is strictly concave.*

The results of this section extend to more general distributions, but at the expense of expositional simplicity.

Proposition 2 characterises the unique equilibrium of the game. It shows that the firm plays a mixed strategy, the equilibrium demand is unit-elastic and the equilibrium profits are the same as in a reference-independent model.

Let $\pi^* = \max_p p(1 - G(p))$ and $p_{ind} = \arg\max_p p(1 - G(p))$ the equilibrium profits and prices of the reference-independent benchmark.

Proposition 2. *There is a unique equilibrium* $\left(F, (Q_v^*)_v\right)$ *where*

- *The firm plays the mixed strategy*

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda}$$

  with $\operatorname{supp} F = [p_{ind}, \overline{p}]$, $\overline{p} \equiv F(\overline{p}) = 1$.

- *If* $p \in \operatorname{supp} F$, *then* $\gamma(V^*(p)) = \frac{\pi^*}{p}$, *i.e., the demand is unit-elastic on the support.*

- *The equilibrium profits are* $\pi^* = \max_p p(1 - G(p))$.

Proof. **Preliminary lemmas:**

The first lemmas guarantee the good behaviour of two equilibrium objects, F, the firm's strategy, and $p^*(v)$ the WTP of each buyer as a function of his type.

Lemma 1. *In any equilibrium,* $p^*(v)$ *is strictly increasing.*

Proof. Take $v_1 < v_2$. Let $p^*(v_1) = p_1$ and $p^*(v_2) = p_2$ be their PPE cutoffs. Assume that $p_1 \geq p_2$. Because p_1, p_2 are PE cutoffs,

$$v_1 - p_1 = -\lambda F(p_1) \tag{1}$$

$$v_2 - p_2 = -\lambda F(p_2) \tag{2}$$

Clearly, $p_1 = p_2$ cannot hold. We either have $v_1 - p_2 \geq -\lambda F(p_2)$ or $v_1 - p_2 < -\lambda F(p_2)$. In the first case, we have

$$v_2 - p_2 > v_1 - p_2 \geq -\lambda F(p_2)$$

contradicting equation (2). In the second case, $F(p_2) < \frac{p_2 - v_1}{\lambda}$ and $p_1 > p_2$ contradict Proposition 1. □

Lemma 2. *Let* F *be an equilibrium strategy. If* $p \in \operatorname{supp} F$, *then there exists* $v \in V$ *such that* $p^*(v) = p$.

Proof. Assume not: there is a $p \in \operatorname{supp} F$ and no v such that $p^*(v) = p$. First, if $p \in \operatorname{supp} F$, then $p^*(v) \geq p$ for some v, for otherwise the firm makes zero profits. The firm can always make strictly positive profits by offering a price $p < \overline{v}$. This would be accepted by all types $v \in [p, \overline{v}]$ because $p^*(v) \geq v$ in any PPE. Because $p^*(\cdot)$ is strictly increasing, this implies that there is a v such that $p^*(\cdot)$ is not continuous at v and $p \in [\lim_{x \searrow v} p^*(x), \lim_{x \nearrow v} p^*(x)]$. By continuity of G and , $\gamma(V^*(p)) = \gamma(V^*(p^*(v)))$. But then both $p^*(v), p \in \operatorname{supp} F$ but they give different profits, a contradiction. □

Lemma 3. *Any equilibrium strategy F is continuous.*

Proof. Assume not. Let $\tilde{p}$ be a point of discontinuity of F. If $\tilde{p}$ is a PE cutoff for some v, then $F(\tilde{p}) = \frac{\tilde{p}-v}{\lambda}$. Using the upper semicontinuity of F and continuity of $\frac{p-v}{\lambda}$, there exists $p' < \tilde{p}$ such that $F(p') < \frac{p'-v}{\lambda}$. By Proposition 1, $\tilde{p}$ cannot be a PPE cutoff of v. This contradicts Lemma 2. □

Lemma 4. *In any equilibrium, $p^*(v)$ is continuous.*

Proof. Assume there exists a point of discontinuity $\tilde{v}$, i.e., $p_1 \equiv \lim_{v \nearrow \tilde{v}} p^*(v) < \lim_{v \searrow \tilde{v}} p^*(v) \equiv p_2$. We have that $F(p^*(v)) = \frac{p^*(v)-v}{\lambda}$ and F is continuous, therefore,

$$F(p_1) = \lim_{v \nearrow \tilde{v}} \frac{p^*(v) - v}{\lambda} < \lim_{v \searrow \tilde{v}} \frac{p^*(v) - v}{\lambda} = F(p_2)$$

We can then find $\tilde{p} \in (p_1, p_2)$ such that $\tilde{p} \in \operatorname{supp} F$ and there exist no v such that $p^*(v) = \tilde{p}$. This contradicts Lemma 2. □

Lemma 3 rules out pure strategies for the firm. It shows that if the firm puts strictly positive mass at one point of the support, it creates a discontinuity in the demand exactly at that point. Then, it wants to take advantage of it.

Lemma 1 and Lemma 4 also imply that we can think of $v^*(p) = \inf\{v : p^*(v) \geq p\}$ as the inverse of $p^*(v)$: for any p in the support, $p^*(v^*(p)) = p$. Furthermore, $V^*(p) = \{v : p^*(v) \geq p\} = [v^*(p), \overline{v}]$ and the demand at any price $\gamma(V^*(p)) = 1 - G(v^*(p))$.

Let $\underline{p} = \min \operatorname{supp} F$ and $\overline{p} = \max \operatorname{supp} F$.

Profits from $\underline{p}$

The profits from $\underline{p}$ are $\underline{p}(1 - G(v^*(\underline{p}))$. Indeed, for any $v \leq p$, $F(v) = 0$. Therefore, $p^*(v) = v + \lambda F(v|v) = v$ is a PE cutoff. This being the smallest PE cutoff possible, it is the PPE cutoff by Proposition 1. Moreover, for any v, $p^*(v) \geq v$. Therefore, all types above $\underline{p}$ accepts it and all types below reject it, i.e., $v^*(\underline{p}) = \underline{p}$. Profits when offering $\underline{p}$ are then $\underline{p}\,(1 - G(\underline{p}))$. These must be the equilibrium profits.

Finding the equilibrium strategy

For any $p \in \operatorname{supp} F$, by indifference on the support,

$$\pi^* \equiv \underline{p}(1 - G(\underline{p})) = p(1 - G(v^*(p)))$$

Therefore,

$$v^*(p) = G^{-1}\left(\frac{p - \pi^*}{p}\right)$$

for all $p \in \operatorname{supp} F$. Since $\frac{p-\pi^*}{p} \in [0, 1)$ for all $p \geq \underline{p}$, the expression above is well-defined. The equilibrium strategy F must guarantee that a PE cutoff of $v^*(p)$ is $p^*(v^*(p)) = p$:

$$v^*(p) - p = -\lambda F(p) \Rightarrow F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda} \tag{3}$$

using that $p^*(v^*(p)) = p$.

Pinning down $\underline{p}$. For any $p < \underline{p}$, $F(p) = 0$. Therefore, $v^*(p) = p - \lambda F(p) = p$. In equilibrium, we must have

$$\pi^* \geq p(1 - G(p))$$

For any $p > \underline{p}$, we have:

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda} \Rightarrow G^{-1}(\frac{p - \pi^*}{p}) = p - \lambda F(p) < p \Leftrightarrow p(1 - G(p)) < \pi^*$$

using that $F(p) > 0$ for $p > \underline{p}$. Therefore, we have $\underline{p} = \arg\max_p \, p(1 - G(p))$.[7]

F is well-defined on the support

I check here that $\frac{p-G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda}$ is a strictly increasing and positive function. For all $p \geq \underline{p}$,

$$p - G^{-1}(\frac{p - \pi^*}{p}) \geq 0 \Leftrightarrow G(p) \geq \frac{p - \pi^*}{p} \Leftrightarrow \pi^* \geq p(1 - G(p))$$

This is satisfied because $\pi^* = \max p(1 - G(p))$.

I now show that for all $p > \underline{p}$, the derivative of F is strictly positive. This follows from

[7] $\underline{p}$ is well-defined by the strict concavity of $p(1 - G(p))$.

1. For $p > \underline{p}$, $\frac{1-G(p)}{p} < g(p)$: by strict concavity of the profit function, the derivative is negative after the maximum.

2. For any $p > \underline{p}$, $p > G^{-1}\left(\frac{p-\pi^*}{p}\right) \geq \underline{p}$: easily shown by rearranging the expression and using that $\pi^* = \underline{p}(1 - G(\underline{p}))$.

Therefore,

$$F'(p) \propto 1 - \frac{\pi^*}{p^2} \frac{1}{g\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)} > 1 - \frac{\pi^*}{p^2} \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right)}{1 - G\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}$$
$$= 1 - \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right)}{p} > 0$$

using fact 1 and 2 for the first inequality and fact 2 for the last.

Pinning down $\overline{p}$. We have to check that there exists a $\overline{p}$, such that $F(\overline{p}) = 1$. To do that, we will check that there exists p such that $F(p) = 1$. Note that $F(\underline{p}) = 0$ and $F(\overline{v} + \lambda) = \frac{\overline{v}+\lambda-G^{-1}\left(\frac{\overline{v}+\lambda-\pi^*}{\overline{v}+\lambda}\right)}{\lambda} > 1$ (rearranging and using that $G(\overline{v}) = 1$). Therefore, by continuity of F, there exists, $\underline{p} < p < \overline{v} + \lambda$ such that $F(p) = 1$. Then, $\overline{p} \equiv F(\overline{p}) = 1$.[8]

Preferred Personal Equilibrium The last step is to check that the PE cutoffs pinned down by equation (3) are PPE cutoffs. The equilibrium strategy has a derivative strictly smaller than $1/\lambda$. Therefore, for each v, $F(p)$ can cross $\frac{p-v}{\lambda}$ only once. The PE pinned down by equation (3) is thus unique. Hence, it is also a PPE cutoff. □

Proposition 2 illustrates how the firm's commitment problem constrains its behaviour and how it can solve it. First, any pure strategy is not credible. Suppose the firm plays a pure strategy p, then all the types $v \geq p$ expect to buy with probability one. This means that their WTPs are bounded away from p. The firm has then a profitable deviation to a higher price.

The firm can play a mixed strategy only if it is indifferent between any price on the support, i.e., the demand is unit-elastic on the support. Therefore, the firm's mixed strategy induces expectations of trading such that the resulting distribution over WTP is unit-elastic.

Moreover, the firm does not benefit from facing buyers with reference-dependent preferences. The buyers whose valuation is below the support know they will only face prices higher than their valuation. Because they play a PPE, they cannot expect to trade and their WTP is equal

[8]Because F is strictly increasing, $\overline{p}$ is well-defined.

to their valuation. Therefore, they behave like players with no attachment effect. When offering the lowest price on the support, only types with valuation above that price accept, exactly like in a reference-independent model. By the indifference condition, these must be the equilibrium profits. However, some types do end up buying the good at a price above their valuations. The exploitation of the buyers' reference-dependent preferences is compensated by higher probability of no trade when offering high prices.

2.1 Complete information

In this subsection, I look at the behaviour of equilibrium objects when the distribution over valuations converges to a singleton and when the valuation is common knowledge.

In what follows, I look at a sequence of games where the only varying primitive is the prior distribution G. Therefore, abusing notation, I will identify a sequence of games with a sequence of prior distributions. Denote $G_i \xrightarrow{\mathcal{D}} \delta_v$ if for $\tilde{v} \neq v$, $G_i(\tilde{v}) \to \mathbb{1}[\tilde{v} \geq v]$, i.e., G_i converges in distribution (or weakly) to δ_v.

Proposition 3. *Take a sequence of games $\{G_i\}_{i=0}^{\infty}$ such that $G_i \xrightarrow{\mathcal{D}} \delta_v$. All other primitives of the model are fixed.*

Then, equilibrium profits converge to v and the firm's equilibrium strategy converges in distribution to $F_\infty(p) = \frac{p-v}{\lambda}$ with $\operatorname{supp} F_\infty = [v, v+\lambda]$.

Moreover, the limit probability of trade is

$$\frac{v}{\lambda} \ln \frac{v+\lambda}{v}$$

and $\lim_{\lambda \to \infty} \frac{v}{\lambda} \ln \frac{v+\lambda}{v} = 0$.

Proof. **Limit distribution and profits:**

Let F_i be the equilibrium strategy given G_i, $\underline{p}_i = \min \operatorname{supp} F_i$, $\overline{p}_i = \max \operatorname{supp} F_i$ and $\pi_i^* = \underline{p}_i(1 - G_i(\underline{p}_i))$. From Proposition 2, $F_i(p) = \frac{p - G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}{\lambda}$ for all $p \in [\underline{p}_i, \overline{p}_i]$. Using that $G_i^{-1}(x) \to v$ for each $x \in (0,1)$, for each $p \in \mathbb{R}$,

$$F_i(p) \to \begin{cases} 0 & if \quad p < v \\ \frac{p-v}{\lambda} & if \quad p \in [v, v+\lambda] \\ 1 & if \quad p > v+\lambda \end{cases}$$

Profits converge to v as $\max_p\, p(1 - G_i(p)) \to v$.

Probability of trade: Fix a distribution G and the firm's strategy F. From Proposition 2, we know that $v^*(p) = G^{-1}\left(\frac{p-\pi^*}{p}\right)$ for all $p \in [\underline{p}, \overline{p}]$. Inverting the expression, we get $p^*(v) = \frac{\pi^*}{1-G(v)}$ for $v \in [\underline{p}, \overline{p} - \lambda]$. The probability of trade for types $v \in [\underline{p}, \overline{p} - \lambda]$ is

$$Q_v^* = \frac{p^*(v) - v}{\lambda} = \frac{1}{\lambda}\left[\frac{\pi^*}{1 - G(v)} - v\right]$$

For $v > \overline{p} - \lambda$, the probability of trade is 1. For $v < \underline{p}$, probability of trade is 0. The ex-ante probability of trade is therefore

$$\int_{\underline{v}}^{\underline{p}} 0\, dG + \int_{\underline{p}}^{\overline{p}-\lambda} \frac{1}{\lambda}\left[\frac{\pi^*}{1 - G(v)} - v\right] dG + \int_{\overline{p}-\lambda}^{\overline{v}} 1\, dG$$
$$= \frac{\pi^*}{\lambda} \ln \frac{1 - G(\underline{p})}{1 - G(\overline{p} - \lambda)} - \int_{\underline{p}}^{\overline{p}-\lambda} \frac{v}{\lambda} dG + G(\overline{v}) - G(\overline{p} - \lambda)$$

Note that

$$F(\overline{p}) = \frac{\overline{p} - G^{-1}\left(\frac{\overline{p}-\pi^*}{\overline{p}}\right)}{\lambda} = 1 \Leftrightarrow G(\overline{p} - \lambda) = \frac{\overline{p} - \underline{p}(1 - G(\underline{p}))}{\underline{p}} \Leftrightarrow \frac{1 - G(\underline{p})}{1 - G(\overline{p} - \lambda)} = \frac{\overline{p}}{\underline{p}}$$

using that $\pi^* = \underline{p}(1 - G(\underline{p}))$ and

$$0 \leq \int_{\underline{p}}^{\overline{p}-\lambda} v dG = [vG(v)]_{\underline{p}}^{\overline{p}-\lambda} - \int_{\underline{p}}^{\overline{p}-\lambda} G(v) dv \leq [vG(v)]_{\underline{p}}^{\overline{p}-\lambda}$$

As $\overline{p} \to v + \lambda$ and $\underline{p} \to v$, the integral converges to

$$\frac{v}{\lambda} \ln \frac{v + \lambda}{v}$$

□

As the distribution of types converges to the singleton v, the firm's strategy converges to a uniform distribution on $[v, v + \lambda]$. The profits, on the other hand, are always equal to the reference-independent benchmark, $\pi^* = v$ in the limit. As the mass of types accumulate on v, the support does not converge to a singleton. Even though the interval of valuations could become arbitrarily small, the interval of potential WTP stays large: any $p \in [v, v + \lambda]$ can be a PPE cutoff. The firm still needs to mix to create the endogenous demand that makes it indifferent on the support. This means that even as we converge to complete information, a

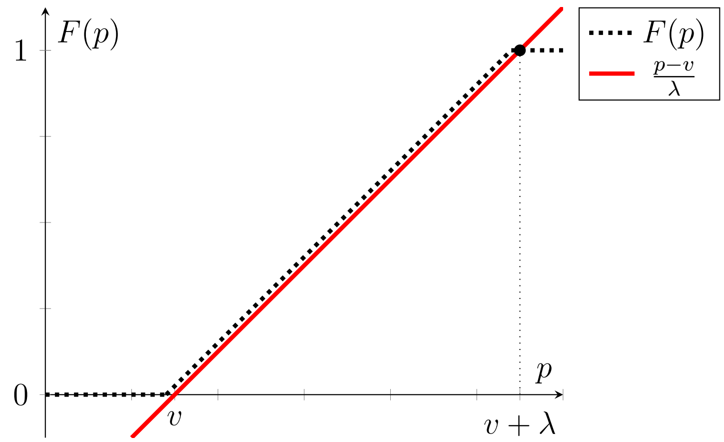

Figure 2: Almost optimal distribution over prices.

large amount of uncertainty is needed to guarantee an equilibrium. This variation leads to a strictly positive probability of no trade in the limit.

Moreover, this probability increases in the attachment effect λ. Intuitively, a higher λ makes the buyer more vulnerable to exploitation but also increases the firm's commitment problem. The firm must lower the probability of trading to compensate for the higher demand induced by a higher λ. This can also be seen from the indifference condition: as profits converge to v for any λ, the higher prices must be compensated for by a higher probability of rejection.

Finally, I note that the limit strategy is arbitrarily close to the one the firm would use if it could commit to a price distribution.

Proposition 4 (Eliaz and Spiegler, 2015)**.** *The solution to the commitment problem:*

$$\sup_{F \in \Delta \mathbb{R}} \int_{-\infty}^{p^*} p \, dF(p)$$

subject to p^* is PPE, is

$$v + \frac{\lambda}{2}$$

and the distribution that attains this profit is $F(p) = \frac{p-v}{\lambda}$.

Proof. See Eliaz and Spiegler (2015). □

The firm chooses a price distribution that maximises its profits amongst all the distribution that implement trade with probability one. By FOSD property of the PPE (Proposition 1), this done by choosing a price distribution as close as possible to $\frac{p-v}{\lambda}$. Note also that this characterisation is different than the one from Heidhues and Kőszegi (2014) because the utility function used is different.

In contrast, if the value of v is common knowledge and there is no commitment, no equilibrium exists.

Proposition 5. *When v is common knowledge and $v > 0$, no equilibrium exists.*

Proof. For any $p^*(v)$, there is a unique best-response of the firm, which is to offer $p^*(v)$. Let p be the equilibrium price, i.e., the equilibrium strategy is $F(\tilde{p}) = \mathbb{1}[\tilde{p} \geq p]$.

If $p \leq v$, then there is a unique PE cutoff $p^*(v) = v + \lambda F(v + \lambda) = v + \lambda$. There is a profitable deviation to $p' = v + \lambda$.

If $p > v$, then $p^*(v) = v + \lambda F(v) = v$ is a PE cutoff and also the smallest PE cutoff. By Proposition 1, it is the PPE cutoff and thus there is no trade in equilibrium. Then, there is a profitable deviation to any $p' \in (0, v]$. □

The key tension is that the firm wants to take advantage of the reference-dependent preferences. However, given the PPE requirement and a deterministic price, the buyer's WTP is only higher than his valuation if the price offered is below his valuation. An equilibrium with no trade is also impossible because any price below the valuation will be accepted for any PPE.

We obtain the non-existence result for the same reason there cannot be a pure strategy equilibrium in the incomplete information environment. Playing a pure strategy shifts the demand if the offer is accepted. Unlike the incomplete information environment, the firm cannot play a mixed strategy because it is facing only one type. Thus, it cannot solve its commitment problem.

Dato et al. (2017) have already observed that in games where players are constrained to play a PPE, an equilibrium does not always exist. They note that with finitely many actions, a PPE strategy never entails mixing. Therefore, if the equilibrium requires mixed strategies, these strategies can never be the players' PPE, even though they could be PEs. Here, the mechanism for non-existence is different as the equilibrium relaxing the PPE constraint would not be in mixed strategies. Instead, it occurs because the buyer's PPE price cutoff is always bounded away from the price offered.

3 Optimal learning strategy

We have so far looked at "extreme" information structures, either complete information or complete lack of information. In this section, I allow the firm to collect additional information on the buyer's valuation by designing an information structure. In Proposition 7, I characterise the firm's preferred information structure and its profits.

I focus on $G \sim U[0,1]$. This restriction is with loss of generality and I discuss below which results are preserved under more general distributions. All functions and sets are assumed to be measurable.[9]

Information structure: Let S be a set of signals and $F : V \to \Delta(S)$ a mapping from types to distributions over signals. An information structure is a pair (F, S). Denote by $F(s|v)$ the distribution of s conditional on v. Abusing notation, $F(v, s)$ is the joint distribution of (v, s) induced by $F(s|v)$ and $G(v)$.

Players' information: The information structure is common knowledge but only the firm observes the signal realisation. The valuation v is still privately known.

Players' strategy: A strategy for the firm is $P : S \to \Delta(\mathbb{R})$, a mapping from signals to distributions over prices.

The PE and PPE are defined in the same way as before. I assume that the buyer forms expectations after having observed his type and the information structure. Given the information structure (F, S) and strategy P, a reference point Q_v is type v's PE:

$$Q_v = \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda Q_v] \, dP(p|s) dF(s|v)$$

and the expected utility of type v given (F, S), P and Q_v is

$$W(v|Q_v) = \int_S \int_{\mathbb{R}} \left(\mathbb{1}[p \leq v + \lambda Q_v](v - p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda Q_v) \right) dP(p|s) dF(s|v)$$

Then, Q_v^* is a PPE if $Q_v^* \in \arg\max_{Q_v \in PE} W(v|Q_v)$.

[9]This implicitly restrict the sets of strategies as I also assume that sets formed endogenously are measurable.

The firm's ex-ante payoffs are

$$\mathbb{E}[\pi(P)|(Q_v^*)_v] = \int_0^1 \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s) dF(s|v) dG(v)$$

Definition 5. *Fix an information structure (F, S). An equilibrium is a profile $(P, (Q_v^*)_v)$ such that $P \in \arg\max \mathbb{E}[\pi(\cdot)|(Q_v^*)_v]$ and for all $v \in [0,1]$, Q_v^* is type v's PPE.*

In equilibrium, the buyer plays according to his PPE based on the information structure and the firm's strategy. The firm best replies to the buyers' PPE based on its information.

I am interested in the firm's preferred information structure. Like in Bergemann and Morris (2016), we can focus on information structures that generate action recommendations. The distribution $\delta_s(p)$ denotes the Dirac measure.

Proposition 6. *Take an information structure (F, S) and an equilibrium $\left(P, (Q_v^*)_v\right)$ in (F, S). Then, there exists an information structure $(\tilde{F}, \cup_{s \in S} \operatorname{supp} P(\cdot|s))$ and an equilibrium $\left(\delta_s, (Q_v^*)_v\right)$ in that information structure such that all players get the same payoffs.*

Proof. See appendix A □

Proposition 6 holds because the only thing that matters for the buyer's PPE is the distribution over prices given his type. Therefore, a standard revelation principle argument holds. If after two different signals, the firm offers the same price, we can modify the information structure to "merge" these two signals. This will not change the distribution over actions and thus all PPEs are preserved.

The firm's problem is

$$\sup_{(F,S)} \int_{V \times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v, s) \tag{4}$$

$$\text{s.t. } p^*(v) = \min\{p : \; v - p = -\lambda\, F(p|v)\}, \tag{5}$$

for all $S' \subseteq S$,

$$\int_{V \times S'} \mathbb{1}[v \in V^*(x)]\, x\, dF(v, x) \geq \int_{V \times S'} \mathbb{1}[v \in V^*(\tilde{P}(x))]\, \tilde{P}(x) dF(v, x) \tag{6}$$

for all $\tilde{P} : S \to \mathbb{R}$

The firm chooses distribution over prices that will determine its profits and the WTP of each type. The constraint (5) pins down the WTP of each buyer. The obedience constraint (6) ensures that the firm is willing to follow almost all price recommendations. I say that an information structure is *admissible* if it satisfies the constraints (6) where the WTP are pinned down by (5). We need to solve for the supremum because the set of constraints is not closed. In particular, a sequence of distributions $F_i(\cdot|v) \to F(\cdot|v)$ that induce $p^*(v) = p'$ for each i can have $p' \neq \min\{p: \ v - p = -\lambda\, F(p|v)\}$.

To fix ideas, I first characterise the firm's first-best solution, i.e., ignoring the obedience constraints. It is equivalent to allow the firm to commit to a distribution over prices, like in Heidhues and Kőszegi (2014). Proposition 4 in the previous section has thus already characterised the first-best solution.

Claim 1. *The first-best solution is to take a sequence* $\{(F_i, S_i)\}$ *such that*

- $F_i(s|v) \to \frac{s-v}{\lambda}$ *with* $\operatorname{supp} F_i \to [v, v+\lambda]$ *for all* v *and* $S_i \to [0, 1+\lambda]$.
- *For each* i, *for each* v, $p^*(v) = v + \lambda$ *and there is trade with probability one.*

I call the distribution $\frac{s-v}{\lambda}$ on $[v, v+\lambda]$ the commitment distribution. The commitment distribution and its modification defined below will be important for the characterisation of the optimal information structure.

Definition 6 (Censored commitment distribution)**.** F *is the cdf of a censored commitment distribution if there exists a* $\tilde{p} \in [v, v+\lambda]$ *and* $\tilde{F}(s|v) < \frac{s-v}{\lambda}$ *such that*

$$F(s|v) = \begin{cases} 0 & if \quad s < v \\ \frac{s-v}{\lambda} & if \quad s \in [v, \tilde{p}] \\ \tilde{F}(s|v) & if \quad s > \tilde{p} \end{cases}$$

The censored commitment distribution behaves like the commitment distribution for prices below $\tilde{p}$ but stays below $\frac{s-v}{\lambda}$ for higher prices. The commitment distribution is a censored commitment distribution with $\tilde{p} = v + \lambda$. See figure 3 for an example.

In the reference-independent model, the firm would perfectly learn the buyer's valuation and offer the valuation. Therefore, profits when $\lambda = 0$ are $\int_V v\, dG(v)$. An information structure is completely noisy if for almost all s and v, $Pr_F[s|v] < 1$ and there is no unique v such that $s \in \operatorname{supp} F(\cdot|v)$.

Proposition 7. *Assume* $v \sim U[0,1]$ *and let* $v^*(s) = \min\{\frac{\int_0^s G(x)dx}{s}, s - \lambda\}$. *The firm's supremum profits are*

$$\pi^* = \int_0^{1+\lambda} \frac{G(s) - v^*(s)}{\lambda} ds$$

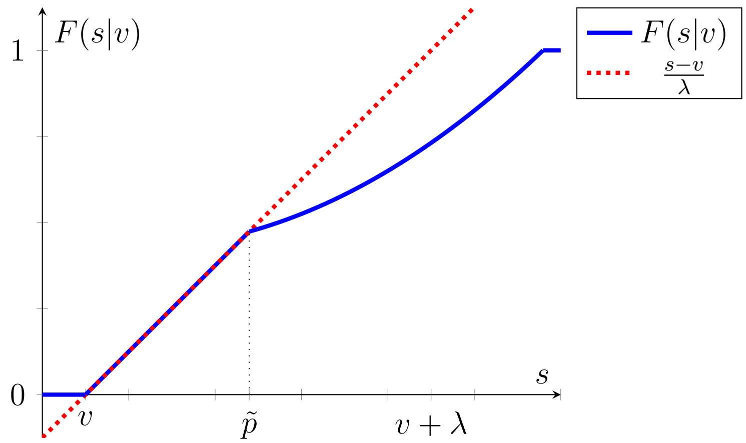

Figure 3: Example of censored commitment distribution

and there exists a sequence of admissible information structure $\{(F_i, S)\}$ that approximate the firm's supremum profits such that

- *For each v, $F_i(\cdot|v)$ converges to a censored commitment distribution.*
- *The sequence $F_i(\cdot|v)$ converges to a completely noisy information structure.*
- *There is downward distortion: lower valuations have lower probability of trading.*

An immediate corollary of Proposition 7 is the comparative statics with respect to λ.

Corollary 1. *Let $v \sim U[0,1]$ and $\pi^*(\lambda)$ be the supremum profits of Proposition 7. Then,*

- $\pi^*(\lambda) > \pi^*(0) = \int_V v\, dG(v)$*, for all* $\lambda > 0$
- $\pi^*(\lambda)$ *is increasing on* $[0, 1/2)$ *and decreasing on* $[1/2, \infty)$
- $\lim_{\lambda \to \infty} \pi^*(\lambda) = \pi^*(0)$

I provide below a sketch of the proof. A complete proof is in appendix B.

Sketch of proof: I start by observing two restrictions on the demand. First, at the lowest signal sent $\underline{s}$, the demand is the same as in the reference-independent model: $\gamma\big(V^*(\underline{s})\big) = 1 - G(\underline{s})$. This condition is similar to the one pinning down profits in Proposition 2: types below the price support only expect prices above their valuation and thus never expect to trade. Therefore, their WTP is equal to their valuation. Second, the demand at any price s is bounded. Indeed, any type v has a bounded WTP: $p^*(v) \leq v + \lambda$. I can show that this implies that for any s, $\gamma\big(V^*(s)\big) \leq 1 - G(s-\lambda)$.

The proof strategy is to find an upper bound on the firm's profits and show that there is a feasible information structure that attains it. The two relaxations I make allows the use of the FOSD property of the PPE. I relax the problem by requiring that obedience constraints hold on intervals of signals $[\underline{s}, s]$ for all s and only upward deviation are considered, $\tilde{P}(x) = x + \epsilon$ for all $\epsilon > 0$,

$$\int_{V\times[\underline{s},s]} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V\times[\underline{s},s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, (x+\epsilon)\, dF(v,x)$$

Restricting attention to upward deviations means that for any $v \in V^*(x+\epsilon)$[10], $x < p^*(v)$. Therefore, we know that $F(x|v) > \frac{x-v}{\lambda}$ by Proposition 1. Integrating over the signals allows to use this FOSD property.

Define $h(s) \equiv \gamma(V^*(s))$, the demand at s. Lemma 6 in the appendix uses the FOSD properties of the PPE to show that the following constraints are further relaxations of the relaxed obedience constraints:

$$\frac{h(s) - h(s+\epsilon)}{\lambda}\, s\,\epsilon + \int_{\underline{s}}^{s} x \frac{h(x) - 1 + G(x)}{\lambda} dx \geq \int_{\underline{s}}^{s} (x+\epsilon)\, \frac{h(x+\epsilon) - 1 + G(x)}{\lambda} dx$$

The demand is non-increasing so the first term is simply a positive term added to the obedience constraint. The next two are the obedience constraints when the firm uses a censored commitment distribution and sets $V^*(x)$ have an interval form: $V^*(x) = [v^*(x), 1]$.

These new constraints only depend on $h(s)$, i.e., the distribution over WTP but not on the exact shape of the distribution F. Thus it is optimal to choose a censored commitment distribution as it gives the highest profits for a given distribution over WTP. We are left with optimising over $h(s)$. Rearranging the constraints and letting $\epsilon \to 0$, we obtain an integral inequality that is easy to solve by integrating by part

$$\int_{\underline{s}}^{s} -x \frac{\partial h(x)}{\partial x} dx \geq \int_{\underline{s}}^{s} h(x) - 1 + G(x) dx$$
$$\Rightarrow\ h(s) \leq \frac{\int_{\underline{s}}^{s} 1 - G(x)dx + \underline{s}h(\underline{s})}{s}$$

Note that nothing guarantees that $h(x)$ is Lipschitz continuous, which was implicitly assumed

[10] Remember that $V^*(s) = \{v : p^*(v) \geq s\}$, the set of types willing to accept s.

above.[11] Indeed, we only know that $h(x)$ is non-increasing as there is almost no structure on $V^*(x)$ beyond $V^*(x) \subseteq V^*(x')$ for $x' < x$. To show that this is without loss, I construct a sequence of relaxed problems where limiting attention to Lipschitz continuous function is without loss and show that we obtain the inequality above in the limit.

The firm wants to set the demand $h(s)$ as high as possible. Using $h(\underline{s}) = 1 - G(\underline{s})$, the inequality above is relaxed by setting $\underline{s} = 0$. Therefore, combining it with the constraint $h(s) \leq 1 - G(s - \lambda)$, which follows from $v^*(s) \geq s - \lambda$, we get an upper bound on profits with $h(s) = \min\{1 - G(s - \lambda), \frac{\int_0^s 1-G(x)dx}{s}\}$ and signal space $S = [0, 1 + \lambda]$. The optimal distribution over WTP takes the form $V^*(s) = [v^*(s), 1] = [\max\{\frac{\int_0^s G(x)dx}{s}, s - \lambda\}, 1]$ and the firm uses a censored commitment distribution for all types. The profits are $\int_0^{1+\lambda} \frac{G(s)-v^*(s)}{\lambda} ds$.

The last step is to show that there exists a sequence of information structure that respects the obedience constraints and converges to the solution of the relaxed problem found above. This is done by taking distributions arbitrarily close to the censored commitment distributions and the derived distribution over WTP while making sure that downward deviations are suboptimal. □

The optimal learning strategy is not to learn the valuation, nor to approximate perfect learning.[12] An imperfect learning strategy uses the two types of uncertainty it generates to credibly exploit the buyer's attachment effect. First, the buyer is uncertain about which signal he generated and therefore which types he is pooled with. At low signals, the firm offers low prices, inducing expectations to buy. Then, at higher signals, the firm offers higher prices, taking advantage of the higher WTP. From the buyer's perspective, he is facing random prices like in Heidhues and Kőszegi (2014). Second, the firm uses the uncertainty it has about the type to credibly offer low prices after a low signal, despite facing some buyers willing to accept higher prices.

However, there are limits to the uncertainty the firm can generate. For example, the firm cannot pool the lowest type in the support with even lower types. Therefore, this type always expects prices higher than his WTP. By the PPE requirement, he must have a WTP equal to his valuation. In turn, this means that he must trade with probability 0. This logic can be extended to more types: relatively low valuations can be pooled with fewer lower types. The

[11]Lipschitz continuity guarantees that (i) we can use the dominated convergence theorem when taking the limit $\epsilon \to 0$ inside the integral on the LHS, (ii) the Fundamental theorem of calculus applies and (iii) $h(s) - h(s + \epsilon) \to 0$.

[12]Remember that there is no equilibrium in the complete information environment, so perfect learning would not respect the obedience constraints.

firm can take advantage of their attachment effect but not fully. Their probability of trade is then smaller than one.

This argument can also be seen from the action recommendation approach. In the first-best solution, each type trades with probability one and therefore $p^*(v) = v + \lambda$. In particular, $p^*(v) \geq 0 + \lambda$ for all v. Moreover, the signal space is $S = [0, 1 + \lambda]$. This solution does not respect the obedience constraints because for any signal in $[0, 0 + \lambda)$, there would be a profitable deviation to $0 + \lambda$. To satisfy the obedience constraints, the firm decreases the WTP of low types to reduce the incentives to increase the price at low signals. But this means that the probability of trading of low types must be less than one. Hence, the firm generates inefficiencies in the form of downward distortion. The optimal way the firm generates the downward distortion is by using censored commitment distributions. These are the distribution over signals that generate the largest profit for a given WTP.

As in the incomplete information environment, the firm generates inefficiencies to maintain the credibility of its own strategy. In both cases, the firm must manage the buyers' expectations to ensure that it is willing to follow its strategy. Unlike the incomplete information environment, the firm can now make more profits than in the reference-independent benchmark. In Proposition 2, the firm can only make the reference-independent profits because types below the support are always expecting prices higher than their valuation. Here, the support of types conditional on the signal is not common knowledge anymore. Therefore, only the lowest type in the prior distribution expects to face prices higher than his valuation.

Finally, the effect of λ on profits is non-monotonic. Increasing λ has two effects. On the one hand, the attachment effect is stronger and therefore, the firm can potentially ask higher prices to the buyers. On the other hand, a higher attachment effect means that the commitment problem of the firm is bigger. It must thus create more inefficiencies to remedy it. Consider a firm implementing its first-best solution: for all v, $p^*(v) = v + \lambda$. Therefore, the firm knows that any type is willing to accept the offer $0 + \lambda$ and thus obedience constraints are not satisfied for any $s \in [0, 0 + \lambda)$. This interval grows larger as λ grows. To satisfy the obedience constraints, the firm must create more inefficiencies by decreasing the demand for a larger interval of types. When λ is greater than $1/2$, the second effect outweighs the first one and profits converge back to the reference-independent benchmark.

The case of non-uniform priors The results derived in this section do not hold for all prior distributions over valuations. The goal of the information structure is to solve the firm's commitment problem while maximising profits. In the discussion of Proposition 7, one such commitment problem was pointed out: if the firm were to implement its first-best strategy, the firm would not want to follow price recommendations in $[0, 0+\lambda)$. However, it could also be that for some distributions, the obedience constraints are not satisfied for higher signals as well. While the first commitment problem exists for all distribution, the second only exists for some. The information structure derived here solves only the first problem.

Note, however, that a result similar to Corollary 1 would hold for any distribution. Indeed, the comparative statics are performed on an upper bound over the profits, which is attainable for some distributions. Therefore, for any prior distribution, the profits from the optimal information structure will converge to the reference-independent benchmark as λ grows large. Similarly, the profits must always be weakly greater than the standard benchmark as these profits are always attainable by creating arbitrarily fine partitions of types and playing the equilibrium of Proposition 2 in each element of the partition.

Public signals Consider a model where the signal realisation is public and the buyer's reference point is set after having observed the signal realisation. In this different environment, the firm's information is common knowledge. Thus, after each signal realisation, we are back to the environment of section 2. The optimal information structure for the firm is then to take an arbitrarily fine partition of the type space and play the equilibrium of Proposition 2 in each element of the partition.

The profits are the same as in the reference-independent benchmark but like in section 2, there is a positive probability of no-trade despite the near-complete information. Another difference is that the WTP is no longer monotonic in the valuation: in each element of the partition $[v, v+\epsilon)$, the support of the mixed strategy is $\approx [v, v+\lambda]$ and the WTP vary on an interval $\approx [v, v+\lambda)$. Therefore, some types might be on higher elements of the partition, but have a lower WTP.

4 Conclusion

In this paper, I study a simple model of monopoly pricing where the buyer has expectation-based reference-dependent preferences, focusing on an attachment effect. The model has two main features. The expectation-based reference point renders the demand an endogenous object. The PPE requirement creates a commitment problem for the firm.

On a theoretical level, this model offers two main lessons. First, uncertainty can help overcome the firm's commitment problem. In all the environments studied, the firm must manage the buyers' expectations, and thus the demand, to maintain a credible strategy. In the incomplete information environment, the firm needs the uncertainty to induce a unit-elastic demand. For its optimal learning strategy, the firm uses the uncertainty to create obedient distributions over prices. While it can deliver equilibrium existence or credible price distributions, using uncertainty necessarily entails inefficiencies. Furthermore, a higher λ, associated with a stronger commitment problem, implies a higher probability of no trade.

The other recurring theme is the impossibility for the firm to exploit the low types. This follows from the buyers' rational expectations as a low type always anticipate prices above his valuation and therefore cannot expect to buy in a PPE. The consequence was particularly stark in the incomplete information model where it made the profits the same as in the reference-independent model. In the optimal learning environment, it generated downward distortions.

The firm responds to the attachment effect by using a random price strategy and the induced inefficiency limits the benefits from facing a reference-dependent buyer. This model thus predicts a random price strategy in a monopolistic environment without any commitment assumption.

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A Proof of Proposition 6

The firm's strategy and information structure as defined are Markov kernels. For simplicity, for any measurable space $(X, \mathcal{X})$, I simply write X. A mapping $Q : X \times Y \to [0,1]$ is a Markov kernel if (i) for any measurable $A \subseteq Y$, $Q(A|\cdot)$ is measurable and (ii) for any $x \in X$, $Q(\cdot|x)$ is a probability measure. I will make repeated use of composition of Markov kernels. Let $Q : X \times Y \to [0,1]$ and $P : Y \times Z \to [0,1]$ be two Markov kernels. Then the composition of $P \circ Q : X \times Z \to [0,1]$ defined as

$$(P \circ Q)(A|x) = \int_Y P(A|y)dQ(y|x) \text{ for all measurable } A \subseteq Z \text{ and } x \in X$$

is a Markov kernel. Furthermore, for all bounded measurable $f : Z \to \mathbb{R}$,

$$\int f(z)d(P \circ Q)(z|x) = \int \int f(z)dP(z|y)dQ(y|x)$$

See e.g., Bauer (1996), chapter VIII, §36.

Proof. Start with an information structure (F, S) and an equilibrium $\big(P, (Q_v^*)_v\big)$. We are going to construct a new information structure $(\tilde{F}, \tilde{S})$ and an equilibrium $\big(\delta_s, (Q_v^*)_v\big)$ such that all players get the same payoffs. We can construct the Markov kernel $\tilde{F} : V \times \tilde{S} \to [0,1]$, where $\tilde{S} = \cup_{s \in S} \operatorname{supp} P(s) \subseteq \mathbb{R}$ as $\tilde{F} = P \circ F$.

Let's first verify that the PPE do not change. Fix a v.

$$\begin{aligned} Q_v &= \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda Q_v] dP(p|s) dF(s|v) \\ &= \int_{\tilde{S}} \mathbb{1}[p \leq v + \lambda Q_v] d\tilde{F}(p|v) \\ &= \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda Q_v] d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) \end{aligned}$$

$$
\begin{aligned}
\text{and } W_{(F,S)}(v|Q_v) &= \int_S \int_{\mathbb{R}} \left(\mathbb{1}[p \leq v + \lambda Q_v](v-p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda Q_v) \right) dP(p|s)dF(s|v) \\
&= \int_{\tilde{S}} \left(\mathbb{1}[p \leq v + \lambda Q_v](v-p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda Q_v) \right) d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \left(\mathbb{1}[p \leq v + \lambda Q_v](v-p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda Q_v) \right) d\delta_{\tilde{s}}(p)d\tilde{F}(\tilde{s}|v) \\
&= W_{(\tilde{F},\tilde{S})}(v|Q_v)
\end{aligned}
$$

This shows that the set of PE and the payoff from each of them does not change under the new information structure and equilibrium. Moving to the firm to the firm's payoffs, we get similarly

$$
\begin{aligned}
\mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] &= \int_0^1 \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s)dF(s|v)dG \\
&= \int_0^1 \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\delta_{\tilde{s}}(p)d\tilde{F}(\tilde{s}|v)dG \\
&= \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v]
\end{aligned}
$$

Note that the integrand is bounded because $\mathbb{1}[v \in V^*(p)] = 0$ for $p > 1 + \lambda$ and offering a negative is a strictly dominated action. The last step is to check that any deviation from $\delta_{\tilde{s}}(p)$ is suboptimal in the new information structure. I show that from any strategy in $(\tilde{F}, \tilde{S})$, we can construct a strategy in (F, S) that yields the same payoff. Let $\tilde{P} : \tilde{S} \times \mathbb{R} \to [0, 1]$ a strategy in $(\tilde{F}, \tilde{S})$. Define the Markov kernel $P' : S \times \tilde{S} \to [0, 1]$ as $P' = \tilde{P} \circ P$. Then,

$$
\begin{aligned}
\mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\tilde{P})|(Q_v^*)_v] &= \int_0^1 \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s})d\tilde{F}(\tilde{s}|v)dG(v) \\
&= \int_0^1 \int_S \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s})dP(\tilde{s}|s)dF(s|v)dG(v) \\
&= \int_0^1 \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP'(p|s)dF(s|v)dG(v) \\
&\leq \mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] = \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v]
\end{aligned}
$$

□

B Proof of Proposition 7

Plan of the proof:

1. Relax the problem by requiring that obedience constraint holds on intervals of signals $[\underline{s}, s]$ for all s and only upward deviation, $\tilde{P}(x) = x + \epsilon$ for all $\epsilon > 0$.

2. Use that $F(s|v) > \frac{s-v}{\lambda}$ for all $s < p^*(v)$ to relax the obedience constraints and make them only depend on $\gamma\big(V^*(s)\big)$, the demand at price s. If the obedience constraints depend only $\gamma\big(V^*(s)\big)$, then it is optimal to choose a censored commitment distribution. We are left with optimising over $\gamma\big(V^*(s)\big)$.

3. Look only at local deviations, i.e., $\epsilon \to 0$, to get an integral inequality that gives some condition on $\gamma\big(V^*(s)\big)$

   (a) Because $\gamma\big(V^*(s)\big)$ is not necessarily Lipschitz continuous, which is needed for the operation described above, I construct a sequence of relaxed problems with a smaller set of relaxed obedience constraints where deviations are bounded away from 0. For each problem, I show that it is without loss to focus on Lipschitz continuous $\gamma\big(V^*(s)\big)$.

   (b) Then, I look at the limit of these problems with Lipschitz continuous $\gamma\big(V^*(s)\big)$ and focusing on the smallest possible deviation in each element of the sequence to derive a condition on $\gamma\big(V^*(s)\big)$.

4. The resulting supremum problem with the condition from local relaxed obedience constraints gives an upper bound on profits.

5. I show that there exists a sequence of information structure respecting the obedience constraints converging to the upper bound.

Proof. Let $\underline{s} = \min S$ and $\overline{s} = \max S$.

Lemma 5. *For any F, $\gamma(V^*(\underline{s})) = 1 - G(\underline{s})$.*

Proof. By definition of PPE and $\underline{s}$, for all $v < \underline{s}$, $p^*(v) = v - \lambda F(v|v) = v < \underline{s}$, using that $F(v|v) = 0$.

On the other hand, $p^*(v) \geq v$, therefore, $p^*(v) \geq \underline{s}$ for all $v \geq \underline{s}$. Thus, $\gamma(V^*(\underline{s})) = 1 - G(\underline{s})$. □

Let $h(s) = \gamma(V^*(s))$, the demand for a given price s. The following lemma states that there exists a relaxation of the original problem where the only constraints are on the demand generated, not on the information structure.

Lemma 6. *The following problem is a relaxation of the firm's problem:*

$$\sup_{S,\, h \in L^1(S)} \int_S x \, \frac{h(x) - 1 + G(x)}{\lambda} dx$$

s.t. for all $s \in S$ *and* $\epsilon > 0$,

$$(h(s) - h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon) - 1 + G(x))dx \tag{7}$$

$$h(s) \in [1 - G(s), 1 - G(s-\lambda)] \text{ for all } s \in S \tag{8}$$

$$h(\underline{s}) = 1 - G(\underline{s});\ h \text{ non-increasing.} \tag{9}$$

Proof. First, focus on the following subset of obedience constraints: for all $s \in S$,

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x)]\, x \, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\,(x+\epsilon)\, dF(v,x) \text{ for all } \epsilon > 0$$

Noting that $V^*(x+\epsilon) \subseteq V^*(x)$, we can rearrange the relaxed obedience constraint as

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x) \setminus V^*(x+\epsilon)]\, x \, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, \epsilon \, dF(v,x)$$

This is equivalent to (see figure 4 for an illustration)

$$\int_{V^*(s) \setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V \setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG$$
$$\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon\, dF(x|v) dG$$

I will now use repeatedly the FOSD property of the PPE: $F(x|v) > \frac{x-v}{\lambda}$ for $x < p^*(v)$ and $F(p^*(v)|v) = \frac{p^*(v)-v}{\lambda}$.

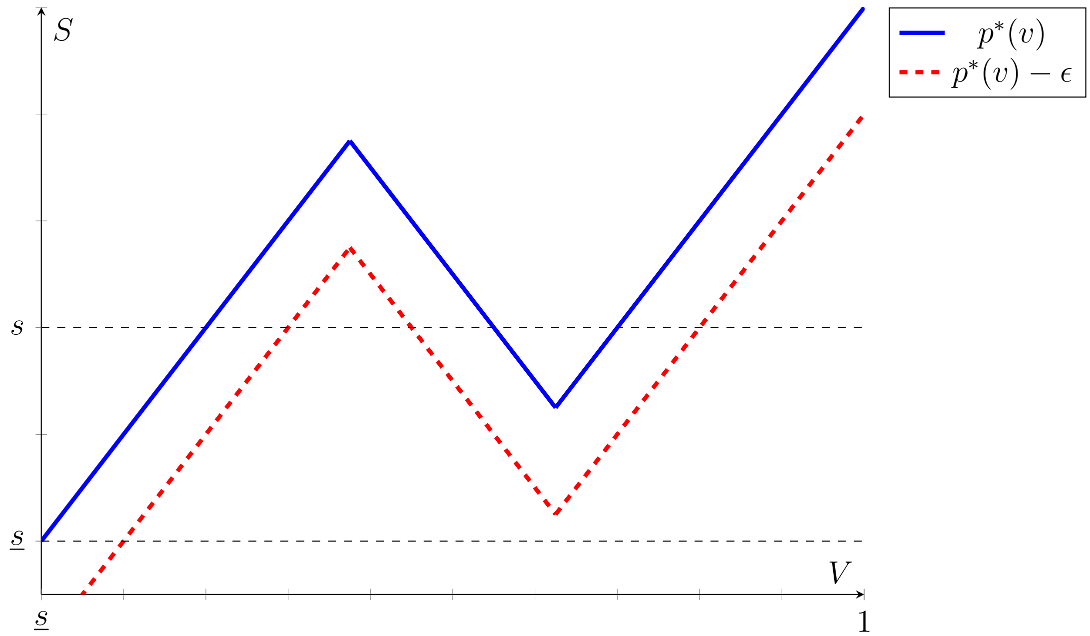

Figure 4: Area of integration with $V^*(x) = \{v : p^*(v) \geq x\}$ and $V^*(x+\epsilon) = \{v : p^*(v)-\epsilon \geq x\}$

Take the RHS first.

$$
\begin{aligned}
&\int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon \, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon \, dF(x|v) dG \\
&\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon \, \mathbb{1}[x \geq v] \frac{1}{\lambda} dx \, dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \mathbb{1}[x \geq v] \epsilon \frac{1}{\lambda} dx \, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x+\epsilon)} \mathbb{1}[x \geq v] \epsilon \frac{1}{\lambda} dG \, dx \\
&\geq \int_{\underline{s}}^{s} \epsilon \, \frac{\gamma(V^*(x+\epsilon)) - 1 + G(x)}{\lambda} dx
\end{aligned}
$$

using the FOSD property on the second line, changing the order of integration in the third and using that $1 \geq \gamma(V^*(x+\epsilon)) + \gamma([0,x]) - \gamma\big(V^*(x+\epsilon) \cap [0,x]\big)$ on the last.

Now focusing on the LHS,

$$
\begin{aligned}
&\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG \\
&\leq \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG}_{I} \\
&+ \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG}_{II}
\end{aligned}
$$

where the inequality simply follows from adding a positive term on the second line.

$$
\begin{aligned}
I &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(F(s|v) - F(p^*(v)-\epsilon|v)\big) dG \\
&\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(\frac{p^*(v)-v}{\lambda} - \frac{p^*(v)-\epsilon-v}{\lambda}\big) dG = \frac{\gamma(V^*(s)) - \gamma(V^*(s+\epsilon))}{\lambda} s\epsilon
\end{aligned}
$$

using that $s \geq x$ and the FOSD property.

$$
\begin{aligned}
II &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} \mathbb{1}[x \geq v] x \frac{1}{\lambda}\, dx\, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x)\setminus V^*(x+\epsilon)} \mathbb{1}[x \geq v] x \frac{1}{\lambda}\, dG\, dx \\
&\leq \int_{\underline{s}}^{s} x \frac{\gamma(V^*(x)) - \gamma(V^*(x+\epsilon))}{\lambda} dx
\end{aligned}
$$

where I use the FOSD property on the first line, change the order of integration on the second and ignore that we must have $\mathbb{1}[x \geq v]$ on the third.

The resulting, relaxed constraint is

$$
\frac{\gamma(V^*(s)) - \gamma(V^*(s+\epsilon))}{\lambda} s\epsilon + \int_{\underline{s}}^{s} x \frac{\gamma(V^*(x)) - \gamma(V^*(x+\epsilon))}{\lambda} dx \geq \int_{\underline{s}}^{s} \epsilon \frac{\gamma(V^*(x+\epsilon)) - 1 + G(x)}{\lambda} dx
$$

Then, remember that the constraint $p^*(v) = \min\{p : v - p = -\lambda F(p|v)\}$ is equivalent to $v - p^*(v) = -\lambda F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) > \frac{p-v}{\lambda}$ (Proposition 1). Relax it by only requiring $v - p^*(v) = -\lambda F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) \geq \frac{p-v}{\lambda}$.

It is now optimal to set $F(s|v) = \frac{s-v}{\lambda}$ for all $s \leq p^*(v)$, i.e., we choose a censored commitment distribution. This operation does not modify the relaxed PPE requirement nor the relaxed obedience constraints but improves profits. Let $h(x) = \gamma(V^*(x))$. The firm's problem becomes

$$\sup_{h\in L^1(S),\, S} \int_S x\,\frac{h(x)-1+G(x)}{\lambda}dx$$

s.t. for all $s \in S$ and $\epsilon > 0$,

$$(h(s)-h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x)-h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon)-1+G(x))dx$$

$$h(s) \in [1-G(s), 1-G(s-\lambda)] \text{ for all } s \in S$$

$$h(\underline{s}) = 1-G(\underline{s});\; h \text{ non-increasing.}$$

where $h(s) \in [1-G(s), 1-G(s-\lambda)]$ comes from $p^*(v) \in [v, v+\lambda]$, $h(\underline{s}) = 1-G(\underline{s})$ follows from Lemma 5 and h non-increasing follows from the definition of $V^*(s)$, i.e., increasing the price necessarily decreases the mass of types willing to accept. □

I am going to do a little detour now and focus on a set of obedience constraints and deviations that are bounded away from zero. Specifically, obedience constraints only need to hold for all $s \in S^i = [\underline{s}+\frac{1}{i}, \overline{s}]$ and $\epsilon \in E^i = [\frac{1}{i}, 1+\lambda]$ for some $i \in \mathbb{N}_0$. Note that $\underline{s} \notin S^i$ and $0 \notin E^i$. Furthermore $S^i \subseteq S^{i+1}$ and $E^i \subseteq E^{i+1}$.

Let K^i be the set of functions satisfying the constraints (7), (8) and (9) for any $s \in S^i$, $\epsilon \in E^i$ and K be the set of functions satisfying these constraints for any $s \in S$ and $\epsilon > 0$. Similarly, define OB^i as the set of functions satisfying the relaxed obedience constraints (7) for any $s \in S^i$ and $\epsilon \in E^i$ and define OB for any $s \in S$ and $\epsilon \in E = [0, 1+\lambda]$. Define

$$\Gamma = \{\phi \in L^1(S) :\ \text{satisfying (8) and (9)}\}$$

where $L^1(S)$ is the set of measurable function from S to $\mathbb{R}$. Note that $K^i = OB^i \cap \Gamma$. Finally, let Lip be the set of Lipschitz continuous functions (not necessarily with the same Lipschitz constant). Endow the spaces defined above with the L^1-norm.

Lemma 7. *Let* $\pi(h) = \int_S \frac{h(x)-1+G(x)}{\lambda}$. *Then,*

$$\sup_{h\in K} \pi(h) \leq \lim_{i\to\infty} \sup_{h\in K^i} \pi(h) = \lim_{i\to\infty} \sup_{h\in K^i\cap Lip} \pi(h)$$

Proof. **1.** $\lim_{i\to\infty} \sup_{h\in K^i} \pi(h)$ exists. This follows from $K^{i+1} \subseteq K^i$, therefore $\sup_{h\in K^{i+1}} \pi(h) \leq \sup_{h\in K^i} \pi(h)$. Moreover, $\sup_{h\in K^i} \pi(h) \geq 0$ as choosing $h(x) = 1-G(x)$ is always possible for any i. Thus, the limit exists.

2. $\lim_{i\to\infty}\sup_{h\in K^i}\pi(h)\geq\sup_{h\in K}\pi(h)$. For each i, $K\subseteq K^i$, therefore, $\sup_{h\in K^i}\pi(h)\geq\sup_{h\in K}\pi(h)$ for each i.

3. $\lim_{i\to\infty}\sup_{h\in K^i}\pi(h)=\lim_{i\to\infty}\sup_{h\in K^i\cap Lip}\pi(h)$

To prove this identity, I will show that $K^i\cap Lip$ is a dense subset of K^i. Because $\pi(h)$ is continuous in h in the L^1-norm, then $\sup_{h\in K^i}\pi(h)=\sup_{h\in K^i\cap Lip}\pi(h)$.

This part is in three steps. Step 1: show that $\Gamma\cap Lip$ is dense in Γ. Step 2: show that int(K^i) is non-empty in Γ. Step 3: Using that Lipschitz continuous functions are dense in int(K^i) because it is open and int$(K^i)\subseteq\Gamma$, and convexity of K^i, show that any function in K^i can be approximated by a function in $K^i\cap Lip$.

Step 1: $\Gamma\cap Lip$ **is dense in** Γ

Take $\phi\in\Gamma$. Define

$$\phi_n(x)=\begin{cases}(1-G(\underline{s}))+\frac{\phi_n(\underline{s}+1/n)-1+G(\underline{s})}{1/n}(x-\underline{s}) & if \quad x\in[\underline{s},\underline{s}+1/n)\\ n\int_{x-1/n}^{x}\phi(z)dz & if \quad x\geq\underline{s}+1/n\end{cases}$$

ϕ_n is differentiable everywhere but at one point, $\underline{s}+1/n$, and its derivative is bounded by n therefore Lipschitz continuous and $\phi_n\in\Gamma$.

We have to show that

$$\lim_{n\to\infty}\int_{\underline{s}}^{\overline{s}}|\phi_n(x)-\phi(x)|dx=0$$

Focusing on $x\geq\underline{s}+1/n$[13].

$$\begin{aligned}&\int_{\underline{s}+1/n}^{\overline{s}}|n\int_{x-1/n}^{x}\phi(z)dz-\phi(x)|dx\\ &\leq\int_{\underline{s}+1/n}^{\overline{s}}n\int_{x-1/n}^{x}|\phi(z)-\phi(x)|dzdx\\ &=\int_{\underline{s}+1/n}^{\overline{s}}n\int_{-1/n}^{0}|\phi(x+y)-\phi(x)|dydx\\ &=\int_{-1/n}^{0}n\int_{\underline{s}+1/n}^{\overline{s}}|\phi(x+y)-\phi(x)|dxdy\\ &\leq\sup\{\int_{\underline{s}+1/n}^{\overline{s}}|\phi(x+y)-\phi(x)|dx\,:\,y\in[-1/n,0]\}\end{aligned}$$

[13] I would like to thank user fourierwho of StackExchange for this proof.

For simplicity, extend the domain to the real line and set $\phi(x) = 0$ when $x \notin [\underline{s}, \overline{s}]$. Let $\psi_m \in C_c(\mathbb{R})$, the set of continuous function in $\mathbb{R}$ with compact support, with $\psi_m \to_{L^1} \phi$. By the Heine-Cantor theorem, any ψ_m is uniformly continuous. We have for all m,

$$\begin{aligned}
&\lim_{y\to 0}\int_{\mathbb{R}} |\phi(x+y)-\phi(x)|dx \\
&\leq \lim_{y\to 0}\int_{\mathbb{R}} |\phi(x+y)-\psi_m(x+y)|dx + \int_{\mathbb{R}} |\psi_m(x+y)-\psi_m(x)|dx + \int_{\mathbb{R}} |\psi_m(x)-\phi(x)|dx \\
&\leq 2\int_{\mathbb{R}} |\psi_m(x)-\phi(x)|dx
\end{aligned}$$

where $\lim_{y\to 0}\int_{\mathbb{R}} |\psi_m(x+y)-\psi_m(x)|dx = 0$ holds because ψ_m is uniformly continuous. Therefore, taking $m \to \infty$, $\lim_{y\to 0}\int_{\mathbb{R}} |\phi(x+y)-\phi(x)|dx = 0$. In turn, it means that $\sup\{\int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y)-\phi(x)|dx \,:\, y \in [-1/n, 0]\} \to 0$ as $n \to \infty$.

Now for $x \in [\underline{s}, \underline{s}+1/n)$, because $|\phi_n(x)|$ and $|\phi(x)|$ are bounded as $n \to \infty$, $\lim_{n\to\infty}\int_{\underline{s}}^{\underline{s}+1/n} |\phi_n(x)-\phi(x)|dx = 0$.

Therefore, $\Gamma \cap Lip$ is dense in Γ.

Step 2: Non-empty interior of $\Gamma \cap OB^i$ in Γ

Take $h(x) = 1 - G(x)$. It is easy to check that $h \in K^i = \Gamma \cap OB^i$. Define $z(s,\epsilon) = \int_{\underline{s}}^{s} h(x) - 1 + G(x)dx - \int_{\underline{s}}^{s}(x+\epsilon)(h(x+\epsilon) - 1 + G(x))dx$ and $\underline{z} = \min_{s,\epsilon} z(s,\epsilon)$. Note that we have $\underline{z} > 0$ because $\underline{s} \notin S^i$ and $0 \notin E^i$.

Now take $\phi(x) \in \Gamma$ with $\int_S |\phi(x) - h(x)|dx \leq \eta$, $\eta > 0$. I will show that

$$(\phi(s) - \phi(s+\epsilon))s\epsilon + \int_{\underline{s}}^{s} x(\phi(x) - 1 + G(x))dx \geq \int_{\underline{s}}^{s} (x+\epsilon)(\phi(x+\epsilon) - 1 + G(x))dx$$

for all $\epsilon \in E^i$, $s \in S^i$ for η sufficiently small. Rearranging the obedience constraint,

$$\begin{aligned}
(\phi(s)-\phi(s+\epsilon))s\epsilon+\int_{\underline{s}}^{s} x(h(x)-1+G(x))dx-\int_{\underline{s}}^{s} (x+\epsilon)(h(x+\epsilon)-1+G(x))dx+\int_{\underline{s}}^{s} x(\phi(x)-h(x))dx \\
\geq \int_{\underline{s}}^{s} (x+\epsilon)(\phi(x+\epsilon) - h(x+\epsilon))dx
\end{aligned}$$

Take the LHS, we have

$$z(s,\epsilon) + (\phi(s) - \phi(s+\epsilon)s\epsilon + \int_{\underline{s}}^{s} x(\phi(x) - h(x))dx \geq \underline{z} - \eta\overline{s}$$

using that ϕ is non-increasing. The RHS gives

$$\int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-h(x+\epsilon))dx \leq \eta(\overline{s}+\epsilon)$$

Therefore, we need

$$\underline{z}-\eta\overline{s} \geq \eta(\overline{s}+\epsilon)$$
$$\underline{z} \geq (2\overline{s}+\epsilon)\eta$$

which holds for all $s \in S^i$ and $\epsilon \in E^i$ for η small enough.

Step 3: $K^i \cap Lip$ is dense in K^i

First observe that int(K^i) is an open set in Γ in the metric space $(\Gamma, L^1\text{-norm})$. Therefore, int$(K^i) \cap Lip$ is dense in int(K^i).

Note that the set K^i is convex. This can be verified by simply summing over the relaxed obedience constraints. The properties of Γ are also maintained when taking convex combinations.

Take some $h \in$ int(K^i). Any function $\phi \in K^i$ can be approximated by a sequence of $\alpha^n h + (1-\alpha^n)\phi$ with the appropriate sequence of α^n. Moreover, any point in the sequence is in the interior of Γ.[14]

Take any $\phi \in K^i$ and $\epsilon > 0$. Let $\phi^n = \alpha^n\phi + (1-\alpha^n)h$, such that $|\phi - \phi^n| < \epsilon/2$ for all $n \geq N$ for some $N \in \mathbb{N}$. Define also $\psi^n \in K^i \cap Lip$ such that $|\psi^n - \phi^n| < \epsilon/2$ for all n, using that $\phi^n \in$ int K^i. Therefore,

$$|\phi - \psi^n| \leq |\phi - \phi^n| + |\phi^n - \psi^n| < \epsilon/2 + \epsilon/2 = \epsilon$$

for $n \geq N$.

Now, given that $\pi(h) = \int_S x\frac{h(x)-1+G(x)}{\lambda}dx$ is continuous in the L^1-norm, we have established that $\sup_{h\in K^i}\pi(h) = \sup_{h\in Lip\cap K^i}\pi(h)$. □

This lemma shows that taking the restricted set of constraints provides another upper bound to our problem. Furthermore, in the restricted problem, it is without loss to restrict attention to Lipschitz continuous functions.

[14]To see this note that there exists $\eta > 0$ such that any $B_\eta(h) \subseteq K^i$. Take $\psi = \alpha h + (1-\alpha)\phi$. I will show that any $w \in B_{\eta\alpha}(\psi)$ is in K^i. First, define $z = h + \frac{w-\psi}{\alpha}$. Then, $|z-h| = |h + \frac{w-\psi}{\alpha} - h| < \alpha\frac{\eta}{\alpha} = \eta$. Therefore $z \in K^i$. Then, choosing $\beta = \alpha$, we have $w = \beta z + (1-\beta)\phi$ and thus $w \in K^i$.

Now, let's focus on $\lim_{i\to\infty}\sup_{h\in K^i\cap Lip}\pi(h)=\pi(h^*)$ (for some h^*). This implies that there exist a sequence $\{h^i\}$ with $h^i\in K^i\cap Lip$ such that $h^i\to_{L^1} h^*$. Let $\underline{\epsilon}_i=\min\ E^i$.

Because each h^i is bounded and of bounded total variation, by Helly's selection theorem, there exists a subsequence $\{h^{i_k}\}$ such that $h^{i_k}(s)\to h^*(s)$ for all $s\in \text{int } S$. Let's focus on that subsequence and rename its elements: $\{h^k\}_{k=0}^{\infty}$. This implies that for each $s\in \text{int } S$, for all $\eta>0$, there exists $P(s,\eta)\in\mathbb{N}$ such that $|h^*(s)-h^k(s)|<\eta$ for all $k\geq P(s,\eta)$ and there exists $M(\eta)\in\mathbb{N}$ such that $\int_S|h^*(s)-h^k(s)|ds<\eta$ for all $k\geq M(\eta)$. Note also that $\int_S|h^i(x)-h^k(x)|dx<\eta$ for all $k,i\geq M(\eta/2)$.

Finally, h^* being the limit of monotone function, it is monotone and thus continuous almost everywhere. Therefore, wherever h^* is continuous, there exists $N(s,\eta)\in\mathbb{N}$ such that $|h^*(s)-h^*(s+\underline{\epsilon}_i)|<\eta$ for all $i\geq N(s,\eta)$.

Fix $\eta>0$ and $s>\underline{s}$ where h^* is continuous. Define $i=\max\{\frac{1}{s},N(s,\eta/3)\}$. Then, for all $k>k^*(s,\eta)\equiv\max\{i,P(s,\eta/3),P(s+\underline{\epsilon}_i,\eta/3)\}$, we have

$$\begin{aligned}|h^k(s)-h^k(s+\underline{\epsilon}_k)|&\leq|h^k(s)-h^k(s+\underline{\epsilon}_i)|\\&\leq|h^k(s)-h^*(s)|+|h^*(s)-h^*(s+\underline{\epsilon}_i)|+|h^*(s+\underline{\epsilon}_i)-h^k(s+\underline{\epsilon}_i)|\\&<\eta/3+\eta/3+\eta/3=\eta\end{aligned}$$

using that $\underline{\epsilon}_i>\underline{\epsilon}_k$ on the first line. Therefore, for all $k>\max\{k^*(s,\eta),M(\eta/2)\}$,

$$(h^k(s)-h^k(s+\underline{\epsilon}_k)s+\int_{\underline{s}}^{s}x(h^k(x)-1+G(x))dx\geq\int_{\underline{s}}^{s}(x+\underline{\epsilon}_k)(h^k(x+\underline{\epsilon}_k)-1+G(x))dx$$
$$\Rightarrow s\eta+\int_{\underline{s}}^{s}x(h^i(x)-1+G(x))dx+s\eta\geq\int_{\underline{s}}^{s}(x+\underline{\epsilon}_k)(h^i(x+\underline{\epsilon}_k)-1+G(x))dx-(s+\underline{\epsilon}_k)\eta$$

We can rearrange the constraint and let $k\to\infty$ (which implies $\underline{\epsilon}_k\to 0$),

$$(3s+\underline{\epsilon}_k)\eta+\int_{\underline{s}}^{s}x\,\frac{h^i(x)-h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}dx\geq\int_{\underline{s}}^{s}h^i(x+\underline{\epsilon}_k)-1+G(x)dx$$
$$\text{letting } k\to\infty,\quad 3s\eta+\int_{\underline{s}}^{s}-x\frac{\partial h^i}{\partial x}dx\geq\int_{\underline{s}}^{s}h^i(x)-1+G(x)dx$$

where we used the dominated convergence theorem, $|h^i|\leq 1$ and $\frac{h^i(x)-h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}$ being bounded by Lipschitz continuity. Integrating by part, we get

$$3s\eta-[h^i(x)x]_{\underline{s}}^{s}+\int_{\underline{s}}^{s}h^i(x)dx\geq\int_{\underline{s}}^{s}h^i(x)dx-\int_{\underline{s}}^{s}1-G(x)dx$$
$$h^i(s)\leq\frac{\int_{\underline{s}}^{s}1-G(x)dx+(1-G(\underline{s}))\underline{s}}{s}+3s\eta$$

Using that $h^i(\underline{s}) = 1 - G(\underline{s})$.

Then, we can take a sequence of $\eta \to 0$, and thus $i \to \infty$, and we get for each s where h^* is continuous

$$h^*(s) = \lim_{\eta \to 0} h^i(s) \leq \lim_{\eta \to 0} \frac{\int_{\underline{s}}^{s} 1 - G(x)dx + (1 - G(\underline{s}))\underline{s}}{s} + 3s\eta = \frac{\int_{\underline{s}}^{s} 1 - G(x)dx + (1 - G(\underline{s}))\underline{s}}{s}$$

This holds for any s where $h^*(s)$ is continuous.

Therefore, we get another upper bound on the firm's problem.

$$\sup_{h \in Lip} \int_{\underline{s}}^{\overline{s}} x \frac{h(x) - 1 + G(x)}{\lambda} dx$$

$$\text{s.t. for all } s \in S' : \; h(s) \leq \frac{\int_{\underline{s}}^{s} 1 - G(x)dx + (1 - G(\underline{s}))\underline{s}}{s}$$

$$h(s) \in [1 - G(s), 1 - G(s - \lambda)]$$

$$h(\underline{s}) = 1 - G(\underline{s})$$

for some $S' \subseteq S$ such that $\mu(S') = \mu(S)$, where $\mu(\cdot)$ is the Lebesgue measure. This is solved by setting $\underline{s} = 0$, $\overline{s} = 1 + \lambda$ and $h(s) = \min\{\frac{\int_0^s 1 - G(x)dx}{s}, \; 1 - G(s - \lambda)\}$. $\square$

Showing the upper bound is achievable

I will now show that there exists a sequence of information structure satisfying the constraints and such that the profits converge to the upper bound.

Lemma 8. *Fix an* $\epsilon > 0$. *There exists an information structure* F_ϵ *such that* $V^*(s) = [v^*(s), 1]$ *with* $v^*(s) = \max\{s - \lambda, \frac{\int_0^s \overline{v}(t)dt}{s}\}$, $\overline{v}(s) \in [G(s), G(s + \frac{\epsilon(1+\lambda)}{\lambda})]$ *and*

$$F_\epsilon(s|v) = \begin{cases} 0 & \text{if} \quad s < \underline{p}(v) \\ \frac{s - \underline{p}(v)}{\lambda + \epsilon} & \text{if} \quad s \in [\underline{p}(v), p^*(v)] \\ \frac{s - v^*(s)}{\lambda} & \text{if} \quad s > p^*(v) \end{cases}$$

Moreover, it satisfies the obedience constraints for ϵ *small enough.*

Proof. **Show the information structure is well-defined** Let $\Lambda = \{p : [0,1] \to [0, 1 + \lambda] : p(v) \geq v, p(0) = 0,$ K-Lipschitz continuous and non-decreasing$\}$ with $K > 1$. Endow that space with the L^1-norm. Construct the mapping $\Phi : \Lambda \to \Lambda$ as follows. Take $p \in \Lambda$ and define $\underline{p}(v)$ as

$$\frac{1}{\lambda + \epsilon} = \frac{\frac{p(v) - v}{\lambda}}{p(v) - \underline{p}(v)} \Leftrightarrow \underline{p}(v) = \frac{\lambda + \epsilon}{\lambda} v - \frac{\epsilon}{\lambda} p(v)$$

Note that if ϵ is small enough then $\underline{p}(v)$ is strictly increasing. Let $\overline{v}(s) = G(\underline{p}^{-1}(s))$ and $\tilde{v}(s) = \frac{\int_0^s \overline{v}(t)dt}{s}$. One can check that $\tilde{v}$ is continuous with $\tilde{v}(0) = 0$ and has its derivative bounded away from zero for any p. We can now define $\Phi : p(v) \to \min\{v+\lambda, \tilde{v}^{-1}(v)\} \in \Lambda$.

We know want to apply Schauder fixed point theorem: Every continuous self-map on a nonempty compact and convex subset of a normed linear space has a fixed point. (Ok, 2007)

The function Φ is a composition of mappings that are continuous in the L^1-norm and is thus continuous. The set Λ is compact because it is of bounded variation, bounded and closed. Thus by Helly's selection theorem any sequence in Λ admits a convergent subsequence that converges in Λ because it is closed. It is also convex. Finally, $L^1([0,1])$ is a normed linear space.

Let $p^*(v)$ be a fixed point of Φ. The information structure is

$$F_\epsilon(s|v) = \begin{cases} 0 & \text{if} \quad s < \underline{p}(v) \\ \frac{s-\underline{p}(v)}{\lambda+\epsilon} & \text{if} \quad s \in [\underline{p}(v), p^*(v)] \\ \frac{s-v^*(s)}{\lambda} & \text{if} \quad s > p^*(v) \end{cases}$$

One can check that $\overline{v}(s) \in [G(s), G(s+\frac{\epsilon(1+\lambda)}{\lambda})]$ because $\underline{p}(v) \in [v - \frac{\epsilon(1+\lambda)}{\lambda}, v]$. Finally, v^* is simply the inverse of p^*.

Show it respects the obedience constraints Let $\pi(s,s') = \int_{v^*(s')}^{\overline{v}(s)} s' f(s|v)dv$. To satisfy the obedience constraints, we must have

$$s \in \arg\max_{s'} \pi(s,s')$$

Let's examine upward and downward deviations separately. First, upward deviation. We can write the profits as $\pi(s,s') = \int_{v^*(s')}^{\overline{v}(s)} \frac{s'}{\lambda+\epsilon} dv$. Taking the derivative with respect s', we get:

$$\frac{\overline{v}(s) - v^*(s')}{\lambda+\epsilon} - (v^*(s'))' \frac{s'}{\lambda+\epsilon}$$

Setting the derivative equal to 0 when $s' = s$, we get $\overline{v}(s) - v^*(s) - s(v^*(s))' = 0$. This is always satisfied when $v^*(s) = \frac{\int_0^s \overline{v}(t)dt}{s}$. When, $v^*(s) = s - \lambda$, we need,

$$\overline{v}(s) - s + \lambda - s \leq 0 \Leftrightarrow \overline{v}(s) + \lambda \leq 2s$$

First, note that $v^*(s) = s - \lambda$ if $s - \lambda \geq \frac{\int_0^s \overline{v}(t)dt}{s} \geq \frac{\int_0^s G(t)dt}{s}$. Therefore, solving $s - \lambda \geq \frac{\int_0^s G(t)dt}{s}$, when $\lambda \leq 1/2$, $s \geq 2\lambda$ and when $\lambda > 1/2$, $s \geq \frac{\lambda+1+\sqrt{\lambda^2+2\lambda-1}}{2}$. Second, note that $\overline{v}(s) \leq s + \frac{\epsilon(1+\lambda)}{\lambda}$. Therefore,

$$\overline{v}(s) + \lambda \leq s + \frac{\epsilon(1+\lambda)}{\lambda} + \lambda \leq 2s$$

which is satisfied for ϵ small enough.

Then observe that for $s' > s$,

$$\frac{\overline{v}(s) - v^*(s')}{\lambda + \epsilon} - (v^*(s'))' \frac{s'}{\lambda + \epsilon} \leq \frac{\overline{v}(s') - v^*(s')}{\lambda + \epsilon} - (v^*(s'))' \frac{s'}{\lambda + \epsilon} \leq 0$$

because $\overline{v}(\cdot)$ is increasing. Therefore, there is no profitable upward deviation.

Now, for downward deviations, assume $v^*(s) > s - \lambda$, we can write the payoffs as

$$\int_{v^*(s)}^{\overline{v}(s)} \frac{s'}{\lambda + \epsilon} dv + \int_{v^*(s')}^{v^*(s)} s' \frac{1 - (v^*(s))'}{\lambda} dv$$

Taking the derivative and evaluating it at $s' = s$, we have

$$\frac{\overline{v}(s) - v^*(s)}{\lambda + \epsilon} - (v^*(s))' s \frac{1 - (v^*(s))'}{\lambda} \geq 0$$
$$\Leftrightarrow \overline{v}(s) - v^*(s) - \frac{\lambda + \epsilon}{\lambda}(1 - (v^*(s))')(v^*(s))' s \geq \overline{v}(s) - v^*(s) - (v^*(s))' s = 0$$

because $\frac{\lambda+\epsilon}{\lambda}(1 - (v^*(s))') \leq 1$ for ϵ small enough for all s because $(v^*(s))'$ is bounded away from 0.

We are then left to check that the profit function is concave when $s' < s$. Take the derivative with respect to s' twice, we get

$$-\frac{1 - (v^*(s))'}{\lambda} (v^*(s'))' - \frac{1 - (v^*(s))'}{\lambda} (v^*(s'))' - \frac{1 - (v^*(s))'}{\lambda} (v^*(s'))'' s' \leq 0$$
$$\Leftrightarrow -2(v^*(s'))' - (v^*(s'))'' s' \leq 0$$

This is always satisfied as $v^*(s)s = \int_0^s \overline{v}(t)dt$. Taking the derivative twice on both sides, we get, $2(v^*(s))' + (v^*(s))'' s = \overline{v}'(s) \geq 0$.

If $v^*(s) = s - \lambda$, there are no profitable downward deviation as $f(s|v) = 0$ for all $v \leq v^*(s)$.

$\square$

As $\epsilon \to 0$, profits converge to the upper bound derived in the previous section.