The Optimal Menu of Tests

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Abstract

A decision-maker can test a privately informed agent prior to making a decision. Instead of choosing the test himself, the decision-maker requires the agent to choose the test from a menu. By offering a menu, the decision-maker can use the agent's choice as an additional source of information. The decision-maker must accept or reject the agent who always wants to be accepted. I show that the decision-maker does not benefit from commitment in this context. Using this result, I show in several economic environments when the decision-maker benefits from offering a choice of tests. When the set of feasible tests contains a most informative test, I provide a necessary and sufficient condition for the introduction of a less informative test in the optimal menu. I also show when the decision-maker benefits from a menu when tests vary in difficulty or types are multi-dimensional.

Keywords: Test design, strategic information transmission, menu of tests

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1 Introduction

In many economic settings, decision-makers (DMs) rely on tests to guide their actions. Universities use admission tests as part of their application process, firms interview job

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candidates before they hire them and regulators test products prior to authorisation. In these examples, the DM is deciding whether to accept an agent and his preferences depend on some private information held by the agent: the ability of the student, the productivity of the candidate or the quality of the product. Ideally, the DM would want to use a fully revealing test, but this is often not feasible and thus the ability to learn from the test outcome alone is limited. However, there is an additional channel the DM can learn from: he can offer a menu of tests and let the agent *choose* which test to take. The DM can then use the agent's choice as an additional source of information.

Depending on the setting, constraints on the testing ability can take many different forms. For instance, a hiring firm is constrained by the amount of time and resources it can allocate to the selection process; most universities have to use externally provided tests for their admission procedures; and medicine regulatory agencies face both technological and ethical constraints when authorising new drugs.

In this paper, I study the DM's optimal design of a menu of tests and provide conditions under which the DM learns from both the test choice and the outcome. I first provide a characterisation of the optimal menu as a max-min problem and show that the DM does not benefit from committing to an acceptance rule for any arbitrary domains of feasible tests. I then apply this result to natural economic environments and provide conditions under which the DM benefits from offering more than one test and conditions under which he does not. I consider the following three alternative structures on the feasible tests: (i) the DM has access to a test more informative than all the others, (ii) tests are ordered by their difficulty, and (iii) each test can identify only one dimension of the agent's private information.

More specifically, I study a DM who has to accept or reject a privately informed agent. While the DM wants to accept a subset of types (the A-types) and reject the others (the R-types), the agent always wants to be accepted. The domain of feasible tests is an exogenously given set of Blackwell experiments. The DM designs a menu of tests, a subset of the feasible tests, from which the agent chooses one. For example, in its admission procedure, a university can let a student decide whether he takes a test or how many reference letters he submits. A regulator can let a pharmaceutical company design the clinical trials when seeking authorisation of a new drug. After observing the choice of test and its result, the DM decides whether or not to accept the agent.

In Theorem 1, I show that regardless of whether the DM is able to commit ex ante to an acceptance rule, the optimal menu and strategies remain the same. Moreover, I provide an optimisation problem based on the model with commitment that delivers equilibrium strategies. This result holds for arbitrary type structures and domains of feasible tests. Theorem 1 follows from a max-min representation of the DM's problem where the maximiser chooses the DM and A-types' strategy and the minimiser chooses the R-types' strategy.

In Section 4, I use Theorem 1 to determine which tests are part of the optimal menu in three natural economic environments. In Section 4.1, I consider a domain of feasible tests containing a dominant one, in the sense of Blackwell's (1953) informativeness order. One example of this environment is a university or a firm considering whether to allow candidates to submit optional material like an interview, a standardised test or recommendation letter when applying.[1]

The first result in that environment is that the dominant test is always part of an optimal menu. In the example above, it means that the university should always allow students to take the admission test or submit the optional reference letter. I then examine the case of binary tests. By interpreting one of the two signals as a high signal, a binary test allows for ranking types by their performance on a test. I introduce a property of tests called *A-types unreliability* that is necessary and sufficient to determine whether a dominated test should be included in the optimal menu. Intuitively, a test is A-types unreliable if enough A-types perform well on the test *and* sufficiently many do not perform well on the test. Conversely, a test is *not* A-types unreliable if, for example, all A-types have a higher performance than all R-types but also if all A-types have an intermediate performance on the test. In these cases, I say that the test is *single-peaked*.

In Proposition 1, I show that if the dominant test is A-types unreliable then it is always optimal to include a dominated test in the optimal menu. In fact, I show that whenever a test is A-types unreliable, it is always optimal to include another test in the menu, whatever that other test might be. On the other hand, if the dominant test is not A-types unreliable, then offering only the dominant test is optimal. Proposition 1 helps rationalise the common practice of making parts of an application process optional: if there is a concern that some good candidates would not perform well on the optional parts, offering a choice could in fact be optimal. By offering a choice, the university allows these good candidates to not submit a test where they perform poorly and are punished for this bad performance. This intuition resonates with the advice given to prospective candidates to not submit optional

[1] Some prominent examples of these practices: the University of Chicago allows candidates to submit an optional letter of recommendation and a video profile (University of Chicago, 2024*a*) or Stanford University offers an optional interview in its application process (Stanford University, 2024).

material if it '*may not highlight a strength [...] or be a great opportunity [...] to share more*' from the University of Chicago (2024*b*) or to favour '*quality over quantity*' when choosing whether to submit an optional recommendation letter (from NYU admission officer Saviano, 2020).

In the case where the tests generate more than two signals, the results extend as follows. If there exists a subset of signals where one R-type is performing better than some A-type but worse than others, there exists a less informative test that is part of the optimal menu for some prior distribution on types. On the other hand, if the environment is one-dimensional, in the sense that all the tests satisfy the monotone likelihood ratio property and the DM wants to accept any type above a threshold, only the most informative test is offered.

A key observation to show that a singleton menu is optimal is that tests ranked by informativeness induce single-crossing preferences over tests for the agent. When the dominant test is single-peaked, the single-crossing property implies that the equilibrium payoffs are unique and correspond to the equilibrium where only the dominant test is chosen. These results extend *unravelling*-type arguments (e.g., Milgrom, 1981, 2008): instead of unravelling to the most revealing evidence, the strategies unravel to the dominant, but possibly noisy, test. The equilibrium forces concentrate choices on the most informative test but potentially reveal less information than if types would choose different tests.

Tests ranked by informativeness are in a sense vertically differentiated. In Section 4.2 and Section 4.3, I introduce two environments with horizontally differentiated tests and get different results. In Section 4.2, I consider an environment where types are ordered on the real line and feasible tests are ordered by their difficulty. For example, a regulator protecting consumer welfare could mandate labelling to inform consumers about a product's characteristics and decide which certifiers, varying by their stringency, a firm can use.[2] To this end, I define and characterise a new notion of difficulty to compare experiments. This notion captures that varying the difficulty of a test changes which types are better identified: a more difficult test is informative when it is passed, because only high types are likely to produce a high grade, but it is less informative when it is failed. I show that offering only one test is optimal. Similarly to the proof of Proposition 1, I show conditions under which tests ordered by difficulty imply single-crossing preferences over test. I use this property to prove that the only equilibria without unravelling are ones where the DM is better off offering only one test.

[2] For example, the EU mandates energy labels on products such as washing machines, light bulbs or tyres. Varying the difficulty could correspond to varying the threshold to get a 'high grade'.

One lesson from Section 4.1 and Section 4.2 is that ordered types can lead to a singleton menu. I then turn to a multidimensional setting. For example, a hiring firm could be guided by two considerations, the candidate's technical and managerial skills, and focus the interview on either dimension. More generally, I assume that the agent's type has two components and each test is informative about only one of them. Offering tests for both dimensions allows A-types that perform badly in one dimension to select the test where they perform best. I show that the optimal menu contains both tests for any prior if and only if the DM wants to accept any type that performs well in at least one dimension. This would be the case if the hiring firm would want to hire a candidate with high technical skills but no managerial skills and vice-versa. On the other hand, if the firm cares about both dimensions simultaneously, then, for some priors, it uses only one test.

Finally, I consider the following extension to the baseline model. I extend the strategy space of the agent by allowing non-verifiable communication and costless unobserved effort. I first extend the max-min characterisation to that environment. Using that result, I explore in a setting similar to Section 4.1 with a Blackwell dominant test, whether communication helps the DM and how communication interacts with unobserved effort. I show that whenever the agent can exert effort, communication alone cannot improve the DM's payoffs, but offering a dominated test can. On the other hand, when the agent cannot influence the outcome of the test, communication can strictly improve the DM's payoffs. The appendix B also considers an extension introducing the possibility of random allocation to tests.

Relation to the literature

In this paper I study a model of information provision by a privately informed agent through the choice of a test. Formally, my model is closest to the mechanism design with evidence literature where an agent can bring verifiable information to influence the allocation in a mechanism (Green and Laffont, 1986; Bull and Watson, 2007; Deneckere and Severinov, 2008). In particular, Glazer and Rubinstein (2006) study a similar model where the agent can present only deterministic evidence about his type. In Section 3, I characterise the optimal mechanism and unlike most of that literature, I allow for arbitrary domains of feasible tests that include nondeterministic tests.[3] Introducing stochastic tests allows me to

[3]Evidence is a particular kind of test that takes a deterministic form: the agent can provide evidence that he belongs to a certain subset of types. Another difference with modelling information with evidence is that not all types can participate in all tests, i.e., they cannot provide evidence they do not have. I discuss the

study how the statistical properties of the tests interact with the strategic choices of players in previously unexplored economic environments, for example environments containing a most informative test and tests ordered by their difficulty. It also leads me to identify new conditions on tests, such as A-types unreliability, to determine the optimal menu. These results rationalise when DMs would offer a choice of test, as in the case of optional items in university admission procedures, and when they would not.

A common theme in this literature is that the DM might not need commitment to the allocation rule when designing the optimal mechanism; see, e.g., Glazer and Rubinstein (2004); Glazer and Rubinstein (2006); Ben-Porath et al. (2019); Ben-Porath et al. (2021). In Theorem 1, in addition to showing that commitment has no value, I also show that equilibrium strategies are the outcome of a max-min problem.[4] This formulation is useful to further characterise the strategies used at the optimum.

Another important focus of the literature on strategic disclosure is finding conditions under which all information is revealed in equilibrium; see, e.g., Grossman (1981), Milgrom (1981) and for more recent contributions, Hagenbach et al. (2014) or Carroll and Egorov (2019). In my model, if full information is possible, it is optimal, but I also characterise the optimal choice of test when full information is not attainable. Relatedly, in Section 4.1, I give conditions under which equilibrium payoffs are equal to the ones from only using the maximally informative test. I therefore provide unravelling-type results in a noisy test setting.

The framework of this paper also encompasses models of communication with verification (e.g., Glazer and Rubinstein, 2004; Carroll and Egorov, 2019). In particular, Silva (2024) studies an environment with similar payoffs where the agent can communicate to a DM that has access to a fixed test. He gives conditions under which there is value of communication or not (see also Weksler and Zik, 2022, for a related model and results). In this paper, the DM has access to a richer testing technology and can let the agent choose which test to take. This allows me to make additional results and also extend the conditions of Silva (2024) to determine when communication can help the DM. The connection to these

relation between these two modelling approaches in more detail in Section 2.2. For examples of mechanism design papers with nondeterministic tests, see Ball and Kattwinkel (2022) and Ben-Porath et al. (2021).

[4] This result depends on the payoff function assumed. Ben-Porath et al. (2019) consider payoff functions that are more general than the ones considered here but restrict attention to normal evidence structures (Bull and Watson, 2007; Lipman and Seppi, 1995). Ben-Porath et al.'s (2021) result on the value of commitment implies this one but they do not provide a characterisation of the equilibrium strategies. See also Sher (2011) and Hart et al. (2017) for results on commitment in mechanism design with evidence using different payoff functions.

models is explicitly modelled and discussed in Section 5.

The other branch of literature my paper relates to is information design without sender commitment. In these papers, the agent and the DM correspond to the sender and the receiver, respectively. In particular, this paper is closer to models characterising receiver-optimal tests where the sender can choose which test to take. In Rosar (2017) and Harbaugh and Rasmusen (2018), the receiver designs a test where a privately informed agent can either take the test, possibly at a cost, or take an uninformative test. In these papers, the receiver flexibly designs a test *given* that the sender has a choice. In my paper, the receiver designs the choice, i.e., the menu, given the restrictions on the feasible tests.

Other papers consider the receiver-optimal design of tests where the sender's action is partially observed or unobserved, e.g., DeMarzo et al. (2019), Deb and Stewart (2018), Perez-Richet and Skreta (2022) or Ball (2021) (note that Perez-Richet and Skreta, 2022, also consider observable actions). In a model of dynamic task allocation, Deb and Stewart (2018) share a similar concern about whether the optimal task should be the most informative. The key strategic friction in their model is that the agent can shirk on the task. In my paper, the design of the optimal menu also has to take into account the strategy of the sender; however, unobservable actions fundamentally change the sender's incentives and thus how information is revealed. I provide a framework and additional results in Section 5 where this possibility is allowed.[5]

Finally, there are numerous papers concerned with the design of tests and selection procedures whose themes overlap with this paper's. In a complementary paper, Dessein et al. (2023) show that when the agent can observe the test results before submitting them, test-optional policies cannot be justified from an informational point of view. In contrast, in this paper, the agent does not know the outcome when choosing the test. Whether the test result is observable depends on the economic application considered. This is the case when test scores are automatically sent to universities,[6] they use interviews, or the content of a reference letter is not revealed to the student. On the other hand, some popular standardised tests like the SAT or ACT do not automatically send test scores to universities. Di Tillio et al. (2021) compare the informativeness of selected data to randomly generated data.

[5]There are also papers studying sender-optimal tests when the sender cannot fully commit to reporting the test correctly, e.g., Nguyen and Tan (2021), Lipnowski et al. (2022) or Koessler and Skreta (2022). In Boleslavsky and Kim (2018) and Perez-Richet et al. (2020), the sender can commit but there is a third agent whose effort determines, respectively, the state of the world or the Blackwell experiment actually performed.

[6]Some examples are the Law National Aptitude, Mathematics Admissions test or the Sixth Term Examination Paper Mathematics tests used for undergraduate admissions in the UK or the Dental Admission Test and Optometry Admission Test in the US.

Specifically, they study under which condition a DM would prefer observing the n highest realisations of a data generating process over n random draws. They apply their results to questions similar to this paper like 'Should a candidate in an admission process select which questions they want to answer or should the examiner select the questions?' As in Dessein et al. (2023), a key modelling difference is that the agent knows the realisation of the test outcome when selecting what to report. Krishna et al. (2022) model the university selection process as a contest and show that coarsening grades can be Pareto efficient. Ely et al. (2021) study the optimal allocation of tests from a restricted set to agents with observable characteristics.

2 Model

There is a DM and an agent. The agent has a type $\theta \in \Theta$, $|\Theta| < \infty$, with a common prior $\mu \in \Delta\Theta$. The prior is always full support unless specified otherwise. The set of types is partitioned in two: $\Theta = A \cup R$, $A \cap R = \emptyset$. The type is private information of the agent. The DM must take an action $a \in \{0, 1\}$, that is, accept or reject. The utilities of the DM and the agent are $v(a, \theta) = a\big(\mathbb{1}[\theta \in A] - \mathbb{1}[\theta \in R]\big)$ and $u(a, \theta) = a$, respectively. That is, the DM wants to accept agents in A and reject agents in R. The agent always wants to be accepted. The analysis is virtually unchanged by allowing for DMs with utility functions of the form $v(a, \theta) = a\nu(\theta)$ for some $\nu : \Theta \to \mathbb{R}$.[7] When the action space is binary and all agents prefer $a = 1$ to $a = 0$, assuming that $v(a, \theta) = a\nu(\theta)$ is without loss of generality (see Ben-Porath et al., 2021, for a discussion).

There is a finite exogenous set of tests $T \subseteq \Pi := \{\pi : \Theta \to \Delta X\}$, where X is some finite signal space. The conditional probabilities of test t are $\pi_t(\cdot|\theta)$. The set T captures the restriction on the DM's testing capacity. He can perform only one test from that set. A menu of tests is a subset of the feasible tests, $\mathcal{M} \subseteq T$.

For each menu $\mathcal{M}$, the timing of the game is as follows.

1. The agent learns his type θ.
2. The agent chooses a test from the menu, the agent's strategy is denoted by $\sigma : \Theta \to \Delta\mathcal{M}$.

[7]To see this, observe that there is implicitly a weight on accepting the type which is the prior μ. If there is an additional weight ν, we can always transform the prior to take into account these extra considerations: $\tilde{\mu}(\theta) = \frac{\mu(\theta)|\nu(\theta)|}{\sum_\theta \mu(\theta)|\nu(\theta)|}$ and conduct the analysis using the prior $\tilde{\mu}$. This has been pointed out by other papers using this kind of preferences, e.g., Glazer and Rubinstein (2004) or Deb and Stewart (2018).

3. A signal x is drawn according to the realised test $\pi_t(\cdot|\theta)$.

4. The DM chooses an action based on the realised test choice and outcome; the acceptance probability is denoted by $\alpha : \mathcal{M} \times X \to [0, 1]$.

The solution concept is weak Perfect Bayesian Equilibrium of the game induced by the menu $\mathcal{M}$. There are generally many equilibria and I focus on the DM-preferred equilibrium.[8] For example, for any menu where two tests are offered, there are always equilibria where all types choose the same test, akin to a babbling equilibrium in a cheap-talk model. This selection is also motivated by the nature of the results in this paper. I focus on possibility and impossibility results: when it is optimal to offer more than one test and when it is not. Looking at the DM-preferred equilibrium allows me to establish when it is *possible* to benefit from offering more than one test and when it is *impossible*.

Notation: For any α, denote the probability of type θ to be accepted in test t by $p_t(\alpha; \theta) \equiv \sum_x \alpha(t, x)\pi_t(x|\theta)$. I also write $(\alpha, \sigma) \in$ wPBE($\mathcal{M}$) if there is a belief $\tilde{\mu}$ such that $(\alpha, \sigma, \tilde{\mu})$ is a weak PBE in the subgame when the menu is $\mathcal{M}$.

The optimal design of menu solves

$$\begin{aligned} V = \max_{\mathcal{M} \subseteq T} \max_{\alpha, \sigma} & \sum_{\theta \in A} \mu(\theta) \sum_{t \in \mathcal{M}} \sigma(t|\theta) p_t(\alpha; \theta) - \sum_{\theta \in R} \mu(\theta) \sum_{t \in \mathcal{M}} \sigma(t|\theta) p_t(\alpha; \theta) \\ \text{s.t. } & (\alpha, \sigma) \in \text{wPBE}(\mathcal{M}) \end{aligned}$$

The inner maximisation problem selects, for a fixed menu, the DM and agent strategy to maximise the DM's payoff, under the constraint that they are equilibrium strategies. The outer maximisation problem selects the best possible menu for the DM. I say that the DM's payoffs are higher in menu $\mathcal{M}$ than in $\mathcal{M}'$ if the DM's payoff in the DM-preferred equilibrium in $\mathcal{M}$ is higher than in $\mathcal{M}'$.

2.1 Example: Opting out of an admission test

Suppose a university uses some test for admission and that there are three types of students: $A = \{A1, A2\}$ and $R = \{R1\}$. Consider the set of feasible tests $T = \{t, \emptyset\}$ where $\emptyset$ is an

[8] The results would be exactly the same if I would take DM-preferred Sequential Equilibrium (Kreps and Wilson, 1982) as my solution concept. I comment on this in more detail in the discussion of Theorem 1.

uninformative test. The test t is described by $X = \{x_0, x_1\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = x_0 \\ 2/3 & \text{if } x = x_1 \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

For the purpose of this example, suppose that $\mu(A1) < \frac{2}{3}\mu(R1) < \mu(A2)$. The lessons from this example can be extended to arbitrary priors.

This example can be interpreted as follows. The test t is a test a university uses to get information about students, like an interview, a reference letter or a standardised test. The signal x_1 represents a high grade and x_0 a low grade. A common concern about these standardised tests or interviews is that they can be too easily gamed by unobserved test preparation or fail to identify good students in some categories of the population (see, e.g., Academic Senate UC, 2020, for a report on standardised tests in the University of California system). Similarly, there is a concern that unequal access to good reference letter writers makes it possible that good students get bad recommendation letters (Todd, 2021; Hiribarren, 2020). The parametrisation of the test t captures these phenomena. While A_2 and R_1 are naturally ordered, in the sense that A_2 is more likely to have a good grade than R_1, A_1 corresponds to a type of student that the university wants to accept but generates a lower grade than R_1. The prior could be obtained based on some other information provided by the student, e.g., after having observed transcripts or socioeconomic background. This parametrisation, however, still captures that a high grade is predictive of a high-quality student, in line with the finding from the education literature that standardised tests are predictive of good educational outcomes even after conditioning on various observables such as socioeconomic background or race. Adding $\emptyset$ to the menu allows the student to opt out from the admission test. Another interpretation of this example is that the test is a recommendation letter and adding $\emptyset$ makes that recommendation letter optional. The parametrisation then captures the possibility that $A1$ is a good student but with no connection to professors or teachers that could write a good letter for them.

When only t is offered: The information structure and prior deliver the following best response when only t is offered,

$$\alpha(x,t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The acceptance probabilities of each type are then

$$p_t(\alpha; R1) = 2/3 \qquad\qquad p_t(\alpha; A1) = 1/2 \qquad\qquad p_t(\alpha; A2) = 1$$

When both t and $\emptyset$ are offered: Consider the equilibrium with the following strategies of the agent:

$$\sigma(\emptyset|R1) = \frac{\mu(A1)}{\mu(R1)} \qquad\qquad \sigma(\emptyset|A1) = 1 \qquad\qquad \sigma(t|A2) = 1$$

The student $R1$ mixes between the two tests, t and $\emptyset$, whereas $A1$ chooses $\emptyset$ with probability one and $A2$ chooses t with probability one. If all types play a pure strategy, it is not possible to maintain an equilibrium where both tests are chosen. If that is the case, there is a test that is chosen only by an A-type and in equilibrium the DM must accept with probability one after any signal in that test. Thus R_1 mixes in equilibrium to make the menu $\{t, \emptyset\}$ credible.

Given the agent's strategy, the DM's strategy after t remains the same as before. When the DM observes $\emptyset$, he is indifferent between accepting and rejecting. By accepting with probability $2/3$, $R1$ is indifferent between $\emptyset$ and t. The resulting acceptance probabilities are

$$\mathbb{E}[p(\alpha; R1)] = 2/3 \qquad\qquad p_\emptyset(\alpha; A1) = 2/3 \qquad\qquad p_t(\alpha; A2) = 1$$

Types $R1$ and $A2$ have the same acceptance probabilities as before but $A1$ is accepted with strictly higher probability. Therefore, allowing to opt out strictly increases the DM's payoffs. Intuitively, type $A1$ is a student that is poorly identified by the admission test. Giving the option of opting out allows that student to differentiate himself from the other types without benefiting the bad student $R1$.

2.2 Discussion

Test restriction: The exogenous set of tests T can capture different constraints on the DM's testing capacity. It could be a purely technological constraint; e.g., when choosing amongst standardised tests, universities can choose only from an exogenously given set of tests from test providers. The constraint can also be on some properties of the tests that can be used; e.g., $T \subset \{\pi : \pi \text{ has the MLRP}\}$. Finally, it could come from a capacity constraint in the information processing/acquisition abilities of the DM: e.g., a limited number of sample sizes a researcher can collect, a maximum number of reference letters a university can process or there could be a cost function associated with each experiment $C : \Pi \to \mathbb{R}$ and a maximum cost the DM can pay $c \in \mathbb{R}$, $T \subset \{\pi : c \geq C(\pi)\}$.

Non-verifiable communication: In the first sections of the paper, I do not explicitly model non-verifiable communication. In many applications, communication can be verified or the agent can be liable for lies: for example, when the agent is a regulated firm or a candidate for employment. In these cases, no information transmission can be pure cheap-talk. It is also motivated by a transparency or accountability constraint on the DM: using cheap talk means that two observationally equivalent candidates, i.e., choosing the same test and having the same outcome, are treated differently. Treating observationally equivalent candidates similarly is in line with requirements that selection procedures or regulatory decisions should be based on objective criteria.[9] I discuss how to integrate non-verifiable communication and its robustness in Section 5.

Effort: The outcome of the test is independent of the agent's action. The model would go unchanged if effort were costless and observable because it could be deterred with off-path beliefs. I explore the possibility for costless and unobservable effort in Section 5. Note also that if signals are ordered and the DM uses a cutoff strategy, as in many natural applications, a reasonable assumption on effort would be that the higher the effort, the likelier a high signal. In this case, the agent would always have an incentive to provide high effort. See Deb and Stewart (2018) and Ball and Kattwinkel (2022) for models that take into account both asymmetric information and moral hazard in a model of testing.

Relation to models with evidence: The model can be interpreted as a generalisation of models with evidence. The idea of these models is that each type is endowed with a

[9]For example, CIPD, the UK's professional body for HR and people development, recommends the use of objective criteria when making hiring decisions (CIPD, 2022). In the EU, regulatory authorities should respect the principle of equal treatment that '*requires that comparable situations must not be treated differently [...] unless such treatment is objectively justified*' (Sia and Gohari, 2024).

set of messages that only a subset of types can send. Formally, an evidence structure is a correspondence $E : \Theta \rightrightarrows M$ for some finite set of messages M. Thus, type θ can send messages only in $E(\theta)$. We can capture these models in the following way. The set of feasible tests has $X = \{x_1, x_0\}$ and for all $m \in M$, $\pi_m(x_1|\theta) = 1 \Leftrightarrow \theta \in E^{-1}(m)$. Thus a test m perfectly reveals whether θ is in $E^{-1}(m)$ or in $\Theta \setminus E^{-1}(m)$. In a model with evidence, a type θ can never reveal he is in $\Theta \setminus E^{-1}(m)$ for a message $m \notin E(\theta)$. However, in the testing model, we can always incentivise any type to not choose such a test by setting $\alpha(x_0, m) = 0$ for all m. This strategy could be justified because (x_0, m) would always be off-path. On the other hand, I do not allow the agent to withhold evidence after seeing its realisation as in Dye (1985).

3 Characterisation of the optimal menu

An important step in the characterisation of the optimal menu is to show that we can use the case where the DM can commit to a strategy to characterise the DM-preferred equilibrium. I also show that in the DM-preferred equilibrium, the A-types play a pure strategy.

Abusing notation, I will abstract momentarily from the choice of menu and let $\sigma : \Theta \to \Delta T$, i.e., the agent's strategy is a choice over all feasible tests and $\alpha : X \times T \to [0, 1]$ the strategy of the DM at any given test. Denote by $\sigma^{\Theta'} = \big(\sigma(\cdot|\theta)\big)_{\theta \in \Theta'}$ for any $\Theta' \subseteq \Theta$ the vector of strategies for a subset of types. Using this notation, σ^A is the vector collecting the strategies of A-types and σ^R the vector of the strategies of R-types. I also use the following notation for the DM's payoffs:

$$v(\alpha, \sigma^A, \sigma^R) \equiv \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta').$$

I call the game with commitment the game where the DM can commit to a decision rule α before observing the choice of test and the outcome. I say that the DM does not benefit from commitment if the payoffs from the DM-preferred equilibrium are the same in the game with commitment and in the original game.

I first observe in the following lemma that the DM's value from the game with commitment is obtained from a max-min problem.

Lemma 1. *The DM's payoff in the DM-preferred equilibrium in the game with commitment* *is*

$$\max_{\alpha,\sigma^A}\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R). \tag{1}$$

All proofs are relegated to appendix A.

To understand Lemma 1 better, first note that we can rewrite (1) as

$$\max_{\alpha}\big[\max_{\sigma^A}\sum_{\theta\in A}\mu(\theta)\sum_{t\in T}\sigma(t|\theta)p_t(\alpha;\theta)-\max_{\sigma^R}\sum_{\theta'\in R}\mu(\theta')\sum_{t\in T}\sigma(t|\theta)p_t(\alpha;\theta')\big].$$

That is, it corresponds to the problem of finding the best strategy α when the agent chooses a test to maximise his payoffs given α. Because the interests of the A-types are fully aligned with the DM's, they try to maximise his payoffs whereas the R-types have opposite interests and thus try to minimise the DM's payoffs. Unlike the original DM problem, the DM can commit to α and thus it does not need to be a best reply.

We can use this characterisation to get the following result for the game without commitment.

Theorem 1. *For any* $(\alpha,\sigma^A)\in\arg\max_{\tilde\alpha,\tilde\sigma^A}\min_{\tilde\sigma^R} v(\tilde\alpha,\tilde\sigma^A,\tilde\sigma^R)$*, an optimal menu is*

$$\mathcal{M}=\cup_{\theta\in A}\operatorname{supp}\sigma(\cdot|\theta).$$

Additionally, for any $\sigma^R\in\arg\min_{\tilde\sigma^R}\max_{\tilde\alpha} v(\tilde\alpha,\sigma^A,\tilde\sigma^R)$*,* $(\alpha,\sigma^A,\sigma^R)$ *are the strategies in the corresponding DM-preferred equilibrium.*

Moreover,

- *The DM does not benefit from commitment.*
- *There exists a DM-preferred equilibrium where* σ^A *is in pure strategies and therefore* $|\mathcal{M}|\leq|A|$.

Theorem 1 provides two important tools to characterise the optimal menu. The first one is to show that there is no value of commitment and to provide a maximisation problem that gives exactly which strategies are used in the equilibrium *without commitment*. This is a powerful tool to test equilibria and to establish whether a test should be part of an optimal menu. Theorem 1 also allows to establish that a test is part of an optimal menu in equilibrium, even when it is not strictly optimal to add this test. This observation will be important in the next section, where I will show that some tests are part of the optimal menu and then rely on their properties to characterise equilibria.

That commitment has no value in this game comes from the max-min structure of the characterisation. The result relies on the fact that for any strategy of the A-types, the DM and the R-types play a zero-sum game. Therefore, if this game has an equilibrium, 'max-min=min-max' and the order of moves does not matter in this game: the DM has the same payoffs if he moves first or last.

This observation also implies that the max-min characterisation can be used to extend the result beyond a set up with finite action and payoffs linear in (mixed) strategies. It is for example straightforward to extend the result to a setting where the DM is constrained to use increasing strategies, i.e., $X \subset \mathbb{R}$ and $x > x' \Rightarrow \alpha(x,t) \geq \alpha(x',t)$. This set is compact and convex in $\mathbb{R}^{|X|} \times \mathbb{R}^{|T|}$ and therefore the minimax theorem applies.[10]

I also note that the possibility of committing to a menu is not important for the result. Indeed, if more tests are offered, the agents can be discouraged from taking them with the right off-path beliefs. This is possible as long as we focus on a DM-preferred equilibrium.

The second tool is to establish that the size of the menu is bounded by the number of A-types. This observation limits the number of tests we need to consider. An immediate corollary is also that if there is only one type the DM would like to accept, an optimal menu is to use only one test. In particular, this result shows that in a binary-state environment, the optimal mechanism uses only one test, no matter what the available set of tests is.

Corollary 1. *Suppose $|A| = 1$. Then for any T, there is an optimal menu that uses only one test.*

Ben-Porath et al. (2021) also show that there is no value of commitment in a setup where preferences take a similar form as here.[11] One advantage of Theorem 1 is that it provides a maximisation problem that delivers equilibrium strategies. Knowing that there is no value of commitment to α would, for example, not be enough to establish that it is without loss to consider pure strategies from A-types[12] or to establish that a test is part of an optimal menu in equilibrium when it is not strictly optimal to add it. Second, as mentioned, the max-min characterisation can be used to extend the result beyond a set up with finite action and payoffs linear in strategies.

[10]As a further proof of concept, in appendix C, I show that the result also extends to a case where the DM is constrained to play a cutoff strategy, i.e., $X \subset \mathbb{R}$ and $\alpha(x',t) > 0 \Rightarrow \alpha(x,t) = 1$ for $x > x'$. The set of cutoff strategies is not convex and I use the existence result of Milgrom and Shannon (1994) relying on monotonicity of best-reply correspondences to show equilibrium existence.

[11]The functional form they identify, semi-aligned preferences, is more general than what I assume here. The proof of Theorem 1 extends to semi-aligned preferences.

[12]This observation can simplify the analysis a lot as the environment gets more complex, e.g., in Section 5 when there is a choice of effort and communication on top the choice of test.

Sequential Equilibrium. The equilibrium derived in Theorem 1 is sustained with off-path beliefs that put probability one on some $\theta \in R$. If the solution concept is DM-preferred Sequential Equilibrium (SE) (Kreps and Wilson, 1982), Theorem 1 would also hold. If all tests have full support, then all signals are on-path and weak PBE and SE coincide. If some tests do not have full support, I can always assume that the trembling of R-types is more likely than the trembling of A-types. Then, the DM's off-path beliefs after the pair (t, x) are that the type is an A-type if the support of A and R-types do not coincide and that the type is an R-type otherwise. This observation guarantees that if an A-type finds it profitable to deviate in the problem without commitment, then he would also find it profitable to deviate in the problem with commitment.

4 Economic environments

4.1 Environments with a Blackwell dominant test

It is common in applications that the DM has access to a most informative test because, for example, the choice is simply between a test and opting out of it. This is the case if universities make some parts of their admission procedures optional: for example, optional interviews, reference letters, tests, etc. Having a most informative test can also come from the structure of the constraints. For example, the DM could have a time budget to conduct an interview. The more time the interview takes, the more informative it is. Another possibility is that the DM can easily make a test less informative by simply not conducting part of the test; e.g., if a test is composed of a series of questions, the DM can ignore some of them.

I will use Blackwell's (1953) notion of informativeness.

Definition 1 (Blackwell (1953))**.** *A test t is more informative than t', $t \succeq t'$, if there is function $\beta : X \times X \to [0,1]$ such that for all $x' \in X$, $\sum_x \beta(x, x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x, x') = 1$.*

I call a test t a dominant test if $t \succeq t'$ for all $t' \in T$. If a test is more informative than another, then in any decision problem, i.e., i.e., a set of actions, a prior over states and a utility function over state and actions, using the more informative test yields a higher expected utility (Blackwell, 1953). A first important fact we will record here is that if there is a most informative test, then it is part of an optimal menu.

Lemma 2. *If there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, then there is an optimal menu that includes t.*

This lemma follows from the max-min characterisation of Theorem 1 and the properties of dominant tests. Indeed, if we find a menu where the dominant test t is not included, we can modify the DM's strategy such that one A-type is accepted with higher probability than the test he is choosing, say t', and all R-types are accepted with a lower probability than in t'. This new strategy is either a deviation or a new maximum in the max-min problem. Note that if we knew only that commitment had no value, we would not be guaranteed that t would be part of an optimal menu.

As we have seen in the admission test example in Section 2.1, it can be optimal to add a strictly less informative test in the optimal menu. I show now when the example of Section 2.1 generalises or does not. That is, I provide conditions under which it is optimal to include a dominated test and conditions under which it isn't.

I focus on a binary signals environment, $X = \{x_0, x_1\}$. I will use two properties of binary signals tests. First, when signals are binary, we can define a complete order over types for each test. In this order, types are ranked by their likelihood of generating signal x_1: $\theta \geq_t \theta' \Leftrightarrow \pi_t(x_1|\theta) \geq \pi_t(x_1|\theta')$. Given that tests are binary, this is equivalent to ordering types by the likelihood ratio, $\frac{\pi(x_1|\theta)}{\pi(x_0|\theta)}$.

The second property I use is that when signals are binary, the DM's strategy is automatically monotonic in the signals, i.e., either $\alpha(x_1, t) \geq \alpha(x_0, t)$ or $\alpha(x_1, t) < \alpha(x_0, t)$. In appendix C, I show how the results of this section extend to environments where these two properties are satisfied, i.e., types can be ranked by their likelihood ratios and strategies are increasing.[13]

The following property will be key to establish when using a dominated test is optimal.

Definition 2 (A-types unreliable test)**.** *Let $X = \{x_0, x_1\}$. The test t is A-types unreliable if the following three conditions are satisfied*

- *A-type heterogeneity: there is $\theta_1, \theta_2 \in A$ and $\theta_3 \in R$ such that*

$$\theta_1 \leq_t \theta_3 \leq_t \theta_2, \tag{U-1}$$

[13]Recall that as argued in the discussion of Theorem 1, the max-min characterisation of the DM-preferred equilibrium also holds when strategies are constrained to be increasing.

- *Imprecise at the bottom: there is $\underline{\theta} \in \Theta$ and $\theta \in A$ with $\theta \leq_t \underline{\theta}$ such that*

$$\pi_t(x_1|\underline{\theta}) \left(\sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta) \right) \geq \sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta) \pi_t(x_1|\theta), \tag{U-2}$$

- *Precise at the top: there is $\overline{\theta} \in \Theta$ and $\theta \in A$ with $\theta \geq_t \overline{\theta}$ such that*

$$\sum_{\theta \in A,\, \theta \geq_t \overline{\theta}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in R,\, \theta \geq_t \overline{\theta}} \mu(\theta) \pi_t(x_1|\theta) \geq \pi_t(x_1|\overline{\theta}) \left(\sum_{\theta \in A,\, \theta \geq_t \overline{\theta}} \mu(\theta) - \sum_{\theta \in R,\, \theta \geq_t \overline{\theta}} \mu(\theta) \right) \tag{U-3}$$

To understand the property A-types unreliability, suppose that we interpret signal x_1 as a high signal and the DM only accepts after a high signal. We can then interpret the order $\geq_t$ as ranking types by their performance on the test.

Condition 'A-types heterogeneity' (U-1) first states that there is one A-type that is performing worse than some R-type while some other A-type performs better. This condition captures that the test might be adapted to some A-types in the sense that they perform well on it while other A-types do poorly.

A-types unreliability also requires that the test is 'Imprecise at the bottom' and 'Precise at the top'. These two conditions are quantitative properties determining the extend to which the A-types perform poorly at the bottom of the performance distribution and well at the top. To see this, observe that the RHS of condition 'Imprecise at the bottom' (U-2) is the DM's payoffs when accepting after signal x_1 from types below $\underline{\theta}$. The LHS is the payoffs of the DM when types at the bottom are pooled together and all send signal x_1 with probability $\pi_t(x_1|\underline{\theta})$. A test is 'Imprecise at the bottom' when the DM is better off having types pooled rather than follow the signal.

Similarly, condition 'Precise at the top' (U-3) states that for types high enough according to $\geq_t$, the test is precise: the DM is better off using the test rather than having them pooled and send signal x_1 with probability $\pi_t(x_1|\overline{\theta})$.

Before turning to the results, I note that two simpler prior-independent properties on t can help determine whether the test is A-types unreliable. I call a test *enclosed* if there is $\theta_1, \theta_2 \in A$ such that $\theta_1 \leq_t \theta \leq_t \theta_2$ for all $\theta \in \Theta$, then the test is A-types unreliable. Intuitively, this holds when both the best and worse performer are A-types. An eclosed test is always A-types unreliable.

I call a test *single-peaked* if there is $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq_t \theta \leq_t \theta_2\}$. A single-peaked test is never A-types unreliable. It occurs in a natural specification where all A-types have a higher performance on the test than any R-type. But a test is also single-peaked when A-types have an intermediate performance on the test or are the worst performers.

Proposition 1. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and that $\theta \neq_t \theta'$ for all $\theta, \theta' \in \Theta$.*

If the test t is A-types unreliable, there is an optimal menu containing some $t' \preceq t$.

If the test t is not A-types unreliable, the singleton menu $\{t\}$ is optimal.

The proposition is not an 'if and only if' statement because the optimal menu is not always unique. So it can be that both a menu containing a dominated test and a menu containing only the dominant test are optimal. This situation occurs, for example, when the dominant test is not informative enough and the DM does not act differently after observing different signals.

In terms of our admission test example, Proposition 1 delivers the following intuition: if the university believes that sufficiently many good candidates will not perform well on an admission test, it will be better off giving a choice of tests to the candidates. This choice will allow these bad performers to avoid being punished by the test.

The key feature of A-types unreliability that delivers Proposition 1 is that different A-types have different performances on the test, in the sense of condition (U-1) where some A-types perform worse than R-types while others perform better. Importantly, it is not that the test is not monotone with respect to types. The reason the test has to be flawed in that way is that it needs to be possible to give at least two different A-types incentives to choose a different test. Conditions 'Imprecise at the bottom' (U-2) and 'Precise at the top' (U-3) are then quantitative conditions that ensure that separation will be optimal for the DM. In fact, whenever condition (U-1), which is prior-free, is satisfied, there is always a prior such that conditions (U-2) and (U-3) are satisfied as well.

A key observation to prove that when test t is not A-types unreliable, the singleton menu $\{t\}$ is optimal, is that $p_t(\alpha; \theta) - p_{t'}(\alpha; \theta)$ is single-crossing in θ, with respect to the order $\geq_t$, for any α. I then use two types of arguments. First, I show that whenever condition (U-1) is not satisfied and some types choose test t, any equilibrium is payoff equivalent to an equilibrium where only t is chosen. The argument is purely game-theoretical.

When (U-2) is not satisfied, I show that for any equilibria where at least two tests are chosen, there is a deviation in the max-min problem by eliminating the test chosen by the lowest types in the order $\geq_t$. The single-crossing property allows for a clean characterisation of any candidate equilibrium and construction of a deviation in the max-min problem. The violation of (U-3) is treated symmetrically.

The prior-free conditions to determine whether a test is A-types unreliable, enclosed and single-peaked, deliver the following corollary.

Corollary 2. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and that $\theta \neq_t \theta'$ for all $\theta, \theta' \in \Theta$.*

The menu $\{t\}$ is optimal for any μ if and only if the test t is single-peaked.

The DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$ for any μ and any $t' \in T$ if and only if the test t is enclosed.

Finally, I note that we can use Proposition 1 to establish when it is optimal to include another test t' in the menu, even when t and t' are not Blackwell comparable. To show Proposition 1, I show that whenever the dominant test is A-types unreliable, it is always optimal to include a completely uninformative test. This shows in fact that there is always a menu reaching higher payoffs (sometimes weakly) than the singleton menu $\{t\}$ for any set of feasible tests. Indeed, because any test t' is Blackwell more informative than the uninformative test, the DM can always benefit from adding t' even if t and t' are not Blackwell comparable. Recall that I denote by Π the set of Blackwell experiments with signal space X: $\Pi = \{\pi : \Theta \to \Delta X\}$.

Corollary 3. *Let $X = \{x_0, x_1\}$. If test t is A-types unreliable, then the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$ for any $t' \in \Pi$.*

The ideas of Proposition 1 can be partially extended to more than two signals without imposing a priori restrictions on the DM strategy.[14] First, if all tests satisfy the monotone likelihood ratio property and the DM wants to accept only types above a threshold, the optimal menu is the one that offers only the dominant test.

Proposition 2. *Suppose $\Theta, X \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$ and all tests in T have full support and the monotone likelihood ratio property: for $\theta > \theta'$,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \text{ is increasing in } x.$$

If there is $t \succeq t'$ for all $t' \in T$, then the menu $\{t\}$ is optimal.

[14]As mentioned, in online appendix D, I extend the result of Proposition 1 in a setting where the DM is constrained to use an increasing strategy.

Again, this result holds by showing a single-crossing difference property on the acceptance probability. Intuitively, the reason is that a more informative test sends relatively higher signals for higher types. So if a low type chooses the most informative test, the higher types must also choose that one. This constraint on strategies prevents any pooling of A-types and R-types on two different tests. Combined with Lemma 2 that guarantees the inclusion of the dominant test, we get our result. This result would hold using a weaker information order like Lehmann (1988) or some weakening of it. The key property delivering the result is the single-crossing condition described above.

If it is possible to find two signals, x, x', two A-types A_1, A_2 and one R-type, R_1 such that $\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}$, then there is a test t' strictly less informative than t and a prior such that offering $\{t, t'\}$ is better for the DM than just offering $\{t\}$.

Proposition 3. *Let t be a test. Suppose there are two signals $x, x' \in X$, types $A_1, A_2 \in A$ and $R_1 \in R$ such that*

$$\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}.$$

There is a prior μ and a test $t' \prec t$ such that the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$.

Intuitively, if we interpret x as a high signal, the A-type A_1 sends relatively low signals. Suppose that the prior is such that, if only t is offered, x is accepted and x' is not. In a sense, it means that in the test t, type R_1 performs better than A_1 on the signals x, x'. It is then beneficial for the DM to include a test that pools signals x, x' together. In that new test, type A_1 can choose the coarsened test where the superior performance of type R_1 is less important than in the original test. The conditions in Proposition 2 and Proposition 3 are the same as in Silva (2024), who studies communication in a model with imperfect verification. I discuss the connection with his results in Section 5.

4.2 Environments with tests ordered by difficulty

In many economic environments, the DM does not necessarily have access to a most informative test but can vary the difficulty to pass a test. This is for example the case for a regulator that can decide how demanding a certification test is. Like in Proposition 1, I show that the optimal menu is a singleton. I first formalise the notion of difficulty of a test as follows.

Definition 3 (Difficulty environment)**.** *An environment is a Difficulty environment if* $\Theta \subset \mathbb{R}$*,* $A = \{\theta : \theta > \overline{\theta}\}$ *for some* $\overline{\theta}$*,* $X = \{x_0, x_1\}$*,* $T \subset \mathbb{R}$*, all tests have full support, satisfy the monotone likelihood ratio property and for all* $t > t'$*, and* $\theta > \theta'$*,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}, \text{ for } x = x_0, x_1.$$

If $t > t'$, I will say that t is harder than t'. Importantly, a harder test is not more informative. To understand this fact better, let $\mu(\cdot|x, t)$ be a posterior belief after observing signal x in test t and let $\succeq_{FOSD}$ denote the first-order stochastic dominance order. The monotone likelihood ratio property implies that $\mu(\cdot|t, x_1) \succeq_{FOSD} \mu(\cdot|t, x_0)$, that is, that a higher signal is 'good news' about the type (Milgrom, 1981). The last property in the definition further implies $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$, and this property is tight as formalised below.

Proposition 4. *Test* t *is harder than* t' *if and only if for* $x = x_0, x_1$*,* $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$ *for all prior* μ *(including non full support).*

The prior in Proposition 4 is allowed to be non full support because the proof of necessity is for the case of a binary type, as in the proof of necessity in Proposition 1 in Milgrom (1981). Intuitively, the characterisation shows that a passing grade shifts beliefs more towards higher types in a harder test and a failing grade shifts beliefs more towards lower types in an easy test. Or put differently, the harder a test is, the more informative it is about the high types when there is a passing grade, whereas an easier test is informative about the low types when the test is failed. This reasoning implies that if two tests are ordered by difficulty, they are generally not ordered by informativeness.[15] Figure 1 illustrates two tests ordered by difficulty graphically. As an example, if $\Theta \subset (0, 1)$ and $\pi_t(x_1|\theta) = \theta^t$, we are in a Difficulty environment.

Proposition 5. *In a Difficulty environment, a singleton menu is optimal.*

Proposition 5 offers another illustration of the difficulty of maintaining a profitable non-singleton menu in equilibrium. Like in the case of failure of (U-1) in Proposition 1, I use game-theoretical arguments to show that a singleton menu is optimal. Here, it is possible to construct an equilibrium where more than one test is chosen in equilibrium. However,

[15] There are knife-edge cases where two tests can be compared both in terms of difficulty and informativeness. This situation occurs when the posterior belief at one signal is the same across tests.

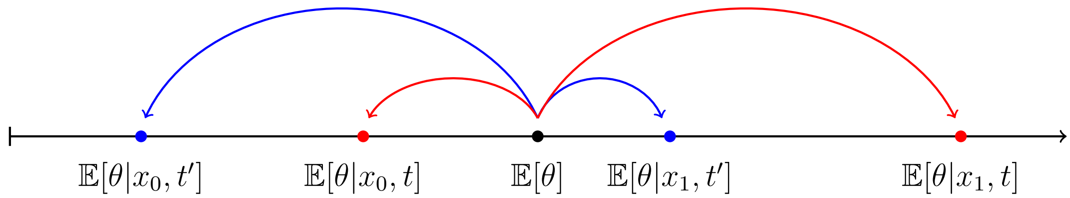


Figure 1: Illustration of posterior means for two tests, $t > t'$. The good news signal x_1 shifts the posterior towards a higher posterior mean in the harder test. The bad news signal x_0 shifts the posterior towards lower posterior mean in the easier test.

the DM strategy needed to sustain that equilibrium is such that he is better off offering only one test.

An important insight in the proof of Proposition 5 is to provide conditions under which a single-crossing property on the choice of tests holds and use it to eliminate potential equilibria. This mirrors the importance of the single-crossing condition in the proof of Proposition 1. The lesson from these two results is that well-ordered types, coupled with the single-crossing property, hurts the DM because it creates 'too much' separation, making a separating equilibrium non-credible.

The proof proceeds in two steps. First, I show that there are at most two tests in an equilibrium menu and if there are two tests, the harder test must be more lenient that the easy test. Intuitively, no one would choose a hard test if on top of that the requirements to be accepted would be harder to fulfil. In particular, I show that after the hard test, the DM must accept with some probability after a failing grade and after the easy test, he must reject with positive probability after a passing grade.

This requirement means that to maintain incentives to select both tests, the DM reacts only to the least informative signal from the test: in the hard test after a failing grade, in the easy test after a passing grade. I then show that in this situation the DM is better off using only one test and reject after a failing grade and accept after a passing grade.

4.3 Bidimensional environments

In this subsection, I apply the tools of Theorem 1 to study environments with bidimensional types. The analysis here can be easily extended to more than two dimensions. I assume that the DM has access to tests that are informative about only one dimension and that the preference of the DM is monotonic along each dimension. I use the following convention to compare elements of $\mathbb{R}^2$: $(\theta_1, \theta_2) \geq (\theta'_1, \theta'_2)$ if $\theta_i \geq \theta'_i$ for $i = 1, 2$.

Definition 4. *An environment is bidimensional if* $\Theta = \Theta_1 \times \Theta_2 \subset \mathbb{R}^2$, $X \subset \mathbb{R}$ *and* $T = \{t_1, t_2\}$ *such that for* $i = 1, 2$,

- *if* $\theta \in A$, *then for all* $\theta' \geq \theta$, $\theta' \in A$
- t_i *has full support and for all* $\theta_i > \theta_i'$,

$$\frac{\pi_{t_i}(x|\theta_i, \theta_j)}{\pi_{t_i}(x|\theta_i', \theta_j)} \text{ is strictly increasing in } x \text{ for any } \theta_j \in \Theta_j$$

- $\pi_{t_i}(x|\theta_i, \theta_j) = \pi_{t_i}(x|\theta_i, \theta_j')$ *for all* $\theta_j, \theta_j' \in \Theta_j$ *and* $x \in X$

This class of test technology is related to Glazer and Rubinstein (2004) and Carroll and Egorov (2019) who also study test allocation in multidimensional environments although the results have a different focus.[16] The first condition captures the idea that a higher type is always better for the DM. The second and third conditions capture the idea that each test is informative about only one dimension and that a higher signal corresponds to a higher type in that dimension. I restrict attention to the case where only two tests, one for each dimension, are available to the DM.

In this environment, whether the DM wants to offer a menu containing both tests depends crucially on his preferences. In particular, I give a necessary and sufficient condition on the preferences such that a full menu is optimal for any prior, i.e., a menu containing both tests. Let $\overline{\theta}_i = \max \Theta_i$.

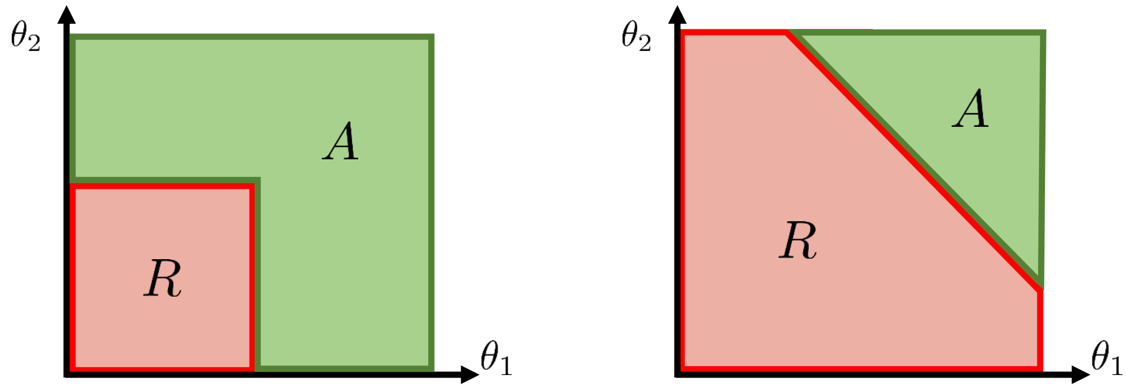
Proposition 6. *Suppose we are in a bidimensional environment. Offering a menu* $\{t_1, t_2\}$ *is strictly optimal for any prior if and only if*

$$\text{for } i = 1, 2,\ (\overline{\theta}_i, \theta_j) \in A, \text{ for all } \theta_j \in \Theta_j. \tag{2}$$

The proof of Proposition 6 works by showing that a deviation from a single test menu is always profitable when condition (2) is satisfied and constructs a prior under which there are no profitable deviations when the condition is not satisfied.

Figure 2 illustrates the condition of Proposition 6 with $\Theta \subset [0, 1]^2$. In Figure 2a, the DM wants the agent's type to be high enough in at least one dimension. Then the DM

[16] Both papers allow for possibly random mechanisms but restrict attention to full revelation of the dimension tested. Glazer and Rubinstein (2004) introduced this environment and characterised the optimal mechanism. Carroll and Egorov (2019) study when the DM can achieve full learning in a setup with more general payoffs. Their condition for full learning is not related to the condition for optimality of having both tests in the menu in Proposition 6.



(a) Full menu always optimal.

(b) Full menu not always optimal.

Figure 2: Illustration of DM's preferences for Proposition 6.

always prefers to offer a full menu to the agent. On the other hand, in Figure 2b, the DM does not want to accept a type that is high in only one dimension. In this case, for some prior, the DM wants to offer only one test. This happens when after any deviation from the singleton menu any A-type is mimicked by too many R-types that cannot be distinguished from him. Finally, note that condition (2) is not related to the complementarity or substitutability of the two dimensions. The key condition is that the highest types in both dimensions are A-types.

5 Extensions and discussions

In this section, I extend the strategy space of the agent by allowing for nonverifiable communication and unobservable costless effort choice. I show that the optimal menu and communication scheme are again the outcome of a max-min problem. I then put more structure on the effort correspondence e and revisit the results of Section 4.1. In appendix B, I consider another extension where I show that the DM benefits from using randomised mechanisms that map a private message of the agent to a possibly random allocation over tests.

I formalise unverifiable communication and unobserved effort choice as follows. Define the effort correspondence as $e : T \times \Theta \rightrightarrows \Theta$ with the restriction that $\theta \in e(t, \theta)$ for all $t \in T$. The set $e(t, \theta)$ denotes the set of types the type θ can copy in test t. If type θ copies θ' in test t, then type θ generates a distribution over signals $\pi_t(\cdot|\theta')$. I call this effort because it will be the leading interpretation once I put more structure on e. The story I have in mind is that the set $e(t,\theta)$ captures the set of types that type θ can copy by putting less effort into the test. Communication is modelled by a finite set C of output messages with $|C| \geq |A|$.

The agent's strategy is now $\sigma : \Theta \to \Delta(T \times C \times \Theta)$ with the restriction that $\sigma(t, c, \theta'|\theta) > 0$ only if $\theta' \in e(t,\theta)$. The payoff of type θ from choosing (t, c, θ') for a given α is $p_{t,c}(\alpha; \theta') = \sum_x \alpha(t, c, x)\pi_t(x|\theta')$. Again, let $\sigma^{\Theta'} = (\sigma(\cdot|\theta))_{\theta\in\Theta'}$. The DM's payoffs become

$$v(\alpha, \sigma^A, \sigma^R) = \sum_{\theta\in A}\sum_{t,x}\sum_{\theta'\in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta)\alpha(t, c, x)\pi_t(x|\theta') - \sum_{\theta\in R}\sum_{t,x}\sum_{\theta'\in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta)\alpha(t, c, x)\pi_t(x|\theta')$$

Note that absent any effort, i.e., $e(t, \theta) = \{\theta\}$ for all types and tests, the original model contains the case of communication. Indeed, we can interpret each pair (t, c) as a distinct test. The set of feasible tests would then be $T' = \{(t, c) : t \in T,\, c \in C\}$ with $\pi_{t,c}(\cdot|\theta) = \pi_t(\cdot|\theta)$. Thus all the results from the previous sections extend to this case.

Let $\operatorname{supp}_T \sigma(\cdot|\theta)$ be the support of a type's strategy restricted to the test choice dimension.

Theorem 2. *For any $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha},\tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$, an optimal menu is*

$$\mathcal{M} = \cup_{\theta\in A} \operatorname{supp}_T \sigma(\cdot|\theta).$$

Additionally, for any $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$, $(\alpha, \sigma^A, \sigma^R)$ are the strategies in the corresponding DM-preferred equilibrium.

Moreover,

- *The DM does not benefit from commitment.*
- *There exists a DM-preferred equilibrium where σ^A is in pure strategies and each $\theta \in A$ sends a different message deterministically, i.e., each A-type truthfully communicates his type.*

The proof follows the same lines as the proof of Theorem 1. Theorem 2 shows that the results extend naturally to an environment where communication and costless unobserved effort are allowed. Because the DM can commit to a strategy, he can always guarantee each A-type at least as much as he would have if he were to pool with another A-type. This argument guarantees that there is a solution to the problem with commitment where any A-type separates from the other A-types.

To illustrate the consequences of unobserved effort and communication, I will focus on a setup with binary tests: $X = \{L, H\}$. As in Section 4.1, we can define for each test t the order $\geq_t$ such that $\theta \geq_t \theta'$ if and only if $\pi_t(H|\theta) \geq \pi_t(H|\theta')$. If the agent can exert unobserved effort, the correspondence e is defined as $e(t, \theta) = \{\theta' : \theta' \leq_t \theta\}$. In the context of a university admission test, it means that any student can achieve the distribution of grades of a worse performing student. If the agent cannot exert unobserved effort, as in the baseline environment, then $e(t, \theta) = \{\theta\}$. In this new setting, signals have an explicit meaning and signals H and L stand for high and low.[17]

An environment, $\mathcal{T}$, is a set of set of feasible tests, i.e., $\mathcal{T} \in 2^{\Pi}$. A menu has *productive communication* if there are two messages c_1, c_2 and a test t such that the pairs $(t, c_1), (t, c_2)$ are on path and $\alpha(\cdot, t, c_1) \neq \alpha(\cdot, t, c_2)$.

Proposition 7. *Let $X = \{H, L\}$. Assume that nonverifiable communication is allowed. Let $\mathcal{T} = \{T :$ there is $t \in T$ with $t \succeq t'$, for all $t' \in T\}$.*

- *If $e(t', \theta) = \{\theta' : \theta' \leq_{t'} \theta\}$ for all $t' \in T$, then a menu with one test and productive communication is never strictly optimal and is sometimes suboptimal in environment $\mathcal{T}$.*

  Let $t \in T$ be such that $t \succeq t'$ for all $t' \in T$. If the test is 'Imprecise at the bottom', i.e., there is $\underline{\theta} \in \Theta$ and $\theta \in A$ with $\theta \leq_t \underline{\theta}$ such that

$$\pi_t(H|\underline{\theta}) \left(\sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta) \right) \geq \sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta)\pi_t(H|\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta)\pi_t(H|\theta),$$

  then there is an optimal menu containing $t' \preceq t$.

- *If $e(t', \theta) = \{\theta\}$ for all $t' \in T$, then a menu with different tests and no productive communication is never strictly optimal in environment $\mathcal{T}$.*

I interpret this result as showing that cheap-talk can be a powerful tool theoretically but that menu design, even using a dominated test, is still likely to play a role in applications.

[17] This way of modelling effort is similar to the modelling choice of Deb and Stewart (2018) and Ball and Kattwinkel (2022). The main difference is that in their setting the agent always has the possibility to get the lowest grade with probability one. Here, the agent must always get the probability distribution of an existing type. The two methods are the same if a type gets the low signal with probability one.

When there is a Blackwell dominant test, cheap-talk, combined with the dominant test, can provide more information than through menu design. However, using cheap-talk is not fully robust to the introduction of effort provision. In any setting where the agent can voluntarily obtain lower grades, cheap-talk alone cannot help the DM but introducing a less informative test can. Below, I revisit the example of Section 2.1 to show the impact of introducing effort and non-verifiable communication. Note also that Proposition 7 establishes that menu design has no value whenever communication is allowed only when there is a Blackwell dominant test.

With effort, condition 'Imprecise at the bottom' (U-2) is sufficient to include a dominated test in the optimal menu. The intuition for this result is the same as the intuition for Proposition 1. If the test is 'Imprecise at the bottom', the DM can introduce a dominated test so that A-types at the bottom of the performance distribution can signal their low ability.

Non-verifiable communication introduces other strategic concerns. Communication is beneficial for the DM if there is productive communication in the DM-preferred equilibrium. If the DM wants at the same time (1) to react to the information in the test and (2) to have different types using different messages, the DM's strategy must be increasing in the test outcome after some messages and must be decreasing after some other message.[18]

These two considerations deliver the first part of Proposition 7. On the one hand, the DM would like to substitute the choice of a dominated test with the choice of the dominant test with a different cheap-talk message as a means of communication. This substitution is in principle possible. But to make it incentive-compatible, the DM creates incentives to reduce effort in the test after some messages, making the test completely uninformative when these messages are used.

The following example revisits the example of Section 2.1 to illustrate Proposition 7. It shows how non-verifiable communication is useless when the agent can obtain a low grade easily. It also provides a situation where a menu consisting of two tests $\{t, t'\}$ with $t \succ t'$ is strictly better than offering $\{t, \emptyset\}$ where $\emptyset$ denotes an uninformative test. An important feature of this example is that the orders over types defined by the tests differ, i.e., $\geq_t \neq \geq_{t'}$. It remains an open question whether including a strictly dominated test can be optimal if $\geq_t = \geq_{t'}$ for all $t, t' \in T$.

Example. Let $\Theta = \{A1, A2, R1\}$ with $A = \{A1, A2\}$. Consider the set of feasible tests

[18] If the DM's strategy is increasing after two different messages, then all types have the same preferred strategy and so they all choose the same message.

$T = \{t, t', \emptyset\}$ where $\emptyset$ is an uninformative test. The prior satisfies $\mu(A1) < \frac{2}{3}\mu(R1) < \mu(A2)$. The test t is described by $X = \{H, L\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = L \\ 1/2 & \text{if } x = H \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = L \\ 2/3 & \text{if } x = H \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 0 & \text{if } x = L \\ 1 & \text{if } x = H \end{cases}$$

Define t' by $\pi_{t'}(H|\theta) = \epsilon\pi_t(H|\theta) + \pi_t(L|\theta)$ with $\epsilon \in (0, 1/2)$.[19] Note that we have $A1 <_t R1 <_t A2$ and $A1 >_{t'} R1 >_{t'} A2$.

We can interpret the example as follows. The DM is a university that cares about technical skills and creative skills and these two skills are anti-correlated: the more creative a type is, the less technical it is. Type $A2$ is a technical type and $A1$ is creative type. The type $R1$ is mediocre in both dimensions. The test t is good at identifying technical types whereas test t' is good at identifying creative types. However, the 'creative' test t' is less precise than the 'technical' test t.

As derived in Section 2.1, the profits from the DM-preferred equilibrium in the case of menus $\{t\}$ and $\{t, \emptyset\}$ are

$$\pi(t) = \mu(A1) \cdot \frac{1}{2} + \mu(A2) \cdot 1 - \mu(R1) \cdot \frac{2}{3},$$
$$\pi(t, \emptyset) = \mu(A1) \cdot \frac{2}{3} + \mu(A2) \cdot 1 - \mu(R1) \cdot \frac{2}{3}.$$

In the case where only t is offered, the DM accepts only accepts after signal H. In the case where the menu $\{t, \emptyset\}$, the DM accepts with probability $2/3$ after observing $\emptyset$ and type $R1$ mixes between t and $\emptyset$. Given that the DM only accepts after a high signal in equilibrium, all types have an incentive to exert full effort in equilibrium.

Now let's consider what would happen when introducing nonverifiable communication. A DM-preferred equilibrium in this case would be one where all types choose t but $A1$ and $A2$ send different messages c_1, c_2. To maintain incentive compatibility, it must be that $R1$ mixes between the two messages, for otherwise one message would fully reveal the type.

[19]The restriction to $(0, 1/2)$ is not essential to the qualitative message of the example but allows us to consider fewer cases in the analysis below.

The DM-preferred equilibrium is described as follows:

$$\sigma(t, c_1|R1) = \frac{3}{4} \cdot \frac{\mu(A1)}{\mu(R1)} \qquad \sigma(t, c_1|A1) = 1 \qquad \sigma(t, c_2|A2) = 1.$$

The DM's strategy is

$$\alpha(x, t, c_1) = \begin{cases} 0 & \text{if } x = L \\ 1 & \text{if } x = H \end{cases} \qquad \alpha(x, t, c_1) = \begin{cases} 1 & \text{if } x = L \\ \frac{1}{2} & \text{if } x = H \end{cases}.$$

In equilibrium, types $A1$ and $A2$ send different messages and $R1$ mixes between the two messages. To benefit from communication, the DM uses a strategy after c_1 that is decreasing in the signal: $\alpha(L, t, c_1) = 1 > \frac{1}{2} = \alpha(H, t, c_1)$. Each of these strategies are best-replies for the players.

The payoffs from this equilibrium is

$$\pi((t, c_1), (t, c_2)) = \mu(A1) \cdot \frac{3}{4} + \mu(A2) \cdot 1 - \mu(R1) \cdot \frac{2}{3} > \pi(t, \emptyset).$$

If we introduce unobserved effort in the equilibrium presented above, then $R1$, when sending message c_1, would have a strict incentive to copy $A2$ to increase his chance to get a low signal L. That would make the test completely uninformative as all types sending c_1 would have the signal distribution of $A1$. By Proposition 7, it is not possible to use communication in a way that incentivise effort *and* have the DM benefit from communication.

However, the DM can do better than the menu $\{t, \emptyset\}$ by offering $\{t, t'\}$. Indeed, consider the following strategies:

$$\sigma(t', R1|R1) = \frac{3}{4} \cdot \frac{\mu(A1)}{\mu(R1)}, \ \sigma(t, R1|R1) = 1 - \frac{3}{4} \cdot \frac{\mu(A1)}{\mu(R1)},$$
$$\sigma(t', A1|A1) = 1, \quad \sigma(t, A2|A2) = 1.$$

The DM's strategy is

$$\alpha(x, t) = \begin{cases} 0 & \text{if } x = L \\ 1 & \text{if } x = H \end{cases} \qquad \alpha(x, t') = \begin{cases} \frac{1-2\epsilon}{2(1-\epsilon)} & \text{if } x = L \\ 1 & \text{if } x = H \end{cases}.$$

In this equilibrium, the creative type $A1$ chooses the 'creative test' and the technical type $A2$ chooses the 'technical test'. The mediocre type mixes between the two options. The DM has a strategy that is always increasing in the signal: $\alpha(H,\tilde{t}) > \alpha(L,\tilde{t})$ for $\tilde{t} = t, t'$ and therefore all types have an incentive to exert effort on the test. The profit from that menu are

$$\pi(t,t') = \mu(A1) \cdot \frac{3}{4} + \mu(A2) \cdot 1 - \mu(R1) \cdot \frac{2}{3} > \pi(t,\emptyset).$$

Relation to verification models with communication. Proposition 7, in the case of $e(t,\theta) = \{\theta\}$, can in fact be extended to arbitrary test restrictions when one test is more informative than all the others. This result derives from Lemma 2 that says that any Blackwell dominant test is part of an optimal menu.

Corollary 4 (to Lemma 2)**.** *Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, nonverifiable communication is allowed and $e(t',\theta) = \{\theta\}$ for all $t' \in T$. Then an optimal menu is $\{t\}$.*

This result tells us that when there is a Blackwell dominant test and no unobserved effort, all the communication can be done through the cheap-talk communication and there is no loss of optimality to only use the dominant test t. At the same time, if the dominant test t is not A-types unreliable, there is no benefit to giving a choice of test (Proposition 1). Thinking of the set of available tests as pairs of test and message, these results taken together show when communication is not needed to reach the optimum, i.e., all types choose the same message-test pair in a DM-preferred equilibrium. Similarly, when the dominant test is A-types unreliable, introducing another message-test pair is optimal, i.e., the dominant test is used in combination with different messages. A similar argument can be made for the conditions in Proposition 2 and Proposition 3 when there are more than two signals.

This discussion shows that the framework of this paper can also accommodate the study of models of communication with verification as in Silva (2024) or Weksler and Zik (2022). In particular, Silva (2024) studies a model with the same payoffs. The condition from Proposition 3 appears in Proposition 2 of his paper and he also notes that monotonicity delivers no value of communication (Proposition 1 in Silva, 2024).[20] Proposition 1 combined with Corollary 4 extends the results of Silva (2024) by giving tight conditions under which

[20] In fact, Proposition 2 from this paper can be derived from Silva's (2024) Proposition 1 in the following way: Any menu with a dominated test can be replicated with copies of the dominant test. We can then apply Silva's (2024) result to show that the DM cannot do better than with only one copy of the dominant test. The proof of Proposition 2 has an independent interest because I show that whenever the dominant test is included with a dominated test, the equilibrium payoffs are unique. It also uses the more permissive concept of accuracy (Lehmann, 1988) and is therefore more general.

the DM benefits communication when the test is binary. Silva (2024) and Weksler and Zik (2022) highlight monotonicity as a key property on the test to determine that communication does not benefit the DM. Instead, I show that *single-peakedness*, a generalisation of monotonicity, is the key sufficient condition to guarantee that communication, or adding a dominated test, does not benefit the DM. Finally, Proposition 7 also shows that communication is vulnerable to unobserved effort in cases where communicating through a choice of test is not.

6 Conclusion

That choice reveals private information is a hallmark observation of economic theory. In this paper, I study how the choice over tests, objects that themselves reveal information, can be leveraged by a decision-maker as an additional source of information. To do so, I show how the statistical properties of the tests interact with the strategic choices of agents and what consequence this interaction has for the design of menus. In particular, I provide conditions under which the decision-maker can benefit from a menu across several environments where tests have different statistical properties. I also show that in several natural environments, the DM cannot do better than using a single test.

The DM faces two constraints to create choice separation: the agent's incentive constraints and his own best-reply constraints. I show in Theorem 1 that this problem is equivalent to solving the DM's problem without best-reply constraints through a max-min characterisation. This characterisation stems from the structure of preferences and is thus applicable in a wide range of mechanism design settings, beyond choices of tests.

Throughout the paper, I provide conditions under which equilibrium forces limit whether the DM can benefit from offering a menu as well as use the max-min characterisation to show when including a test in the menu is optimal. An important technical observation is that single-crossing conditions on the acceptance probability play a key role in having a singleton menu. While single-crossing conditions are usually used to maintain separation in signalling and screening models, in this case separation reveals too much information through the choice. This separation in turn makes it impossible to maintain the incentives to separate in the first place.

Finally, putting more structure on the feasible tests forces us to think about what are reasonable and economically meaningful restrictions on tests. I have considered three possibilities in this paper and introduced a new order on experiments to characterise the notion of difficulty. Developing more ways of restricting feasible tests would allow to explore more trade-offs between experiments and would further enrich the literature on test design.

A Appendix: Omitted proofs

A.1 Proofs of Lemma 1 and Theorem 1

Lemma 1: The problem the DM needs to solve when committing to α is

$$\max_{\alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \tilde{\sigma}(t|\theta') p_t(\alpha;\theta')$$
$$\text{s.t. } \tilde{\sigma}(\cdot|\theta) \in \arg\max \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta), \quad \text{for all } \theta \in \Theta$$

Therefore, $\sum_{t\in T} \tilde{\sigma}(t|\theta)p_t(\alpha;\theta) = \max_{\sigma(\cdot|\theta)} \sum_t \sigma(t|\theta)p_t(\alpha;\theta)$ for all θ. We can plug this expression in the DM's maximisation problem to obtain

$$\max_{\alpha} \max_{\sigma^A} \min_{\sigma^R} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta') p_t(\alpha;\theta')$$

where the $\min$ is obtained because of the minus sign.

Theorem 1: Let $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha},\tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ and

$$\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R).$$

Note that σ^A is fixed in the min-max problem. I will now show that these strategies are equilibrium strategies.

Because the order of maximisation does not matter, $\sigma^A \in \arg\max\min v(\alpha, \tilde{\sigma}^A, \tilde{\sigma}^R)$. Moreover, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R) \Leftrightarrow \sum_{t\in T} \sigma(t|\theta)p_t(\alpha;\theta) \geq \sum_{t\in T} \tilde{\sigma}(t|\theta)p_t(\alpha;\theta)$ for all $\theta \in A$ and $\tilde{\sigma}^A$. This last expression does not depend σ^R. Therefore, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R)$.

Similarly, $\alpha \in \arg\max\min v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Because v is linear in both α and σ^R, (α, σ^R) is a saddle-point of $v(\cdot, \sigma^A, \cdot)$ by the minimax theorem. As for σ^A, $\sigma^R \in \arg\min v(\alpha, \sigma^A, \tilde{\sigma}^R) \Leftrightarrow \sum_{t\in T} \sigma(t|\theta)p_t(\alpha;\theta) \geq \sum_{t\in T} \tilde{\sigma}(t|\theta)p_t(\alpha;\theta)$ for all $\theta \in R$ and $\tilde{\sigma}^R$. Therefore all strategies are best-reply.

Beliefs on-path are formed using Bayes' rule and off-path beliefs are chosen to justify the off-path actions of α (these beliefs are always possible to find as any action is best-reply to some beliefs).

Note also that we can without loss of generality take σ^A to be in pure strategy as it is maximises a linear function.

A.2 Proof of Lemma 2

Take $t \succeq t'$. From Blackwell (1953), we know that the set of distribution of actions conditional on the type under t contains the one under t'. Therefore, for any $\alpha(\cdot, t')$, there is $\alpha(\cdot, t)$ such that

$$\text{for all } \theta \in A, \quad \sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta),$$
$$\text{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta').$$

With this observation in hand, we can prove our result. Suppose that t is not part of the optimal menu. Thus we can find an solution of the max-min problem, $(\alpha, \sigma^A) \in \arg\max_{\alpha', \sigma^{A'}} \min_{\sigma^R} v(\alpha', \sigma^{A'}, \sigma^R)$ with $\sigma(t|\theta) = 0$ for all $\theta \in A$. Take a test t' used in the solution by some $\theta \in A$. Then, we can construct a $(\tilde{\alpha}, \tilde{\sigma}^A)$ such that $p_t(\tilde{\alpha}; \theta) \geq p_{t'}(\alpha; \theta)$, $p_t(\tilde{\alpha}; \theta') \leq p_{t'}(\alpha; \theta')$ for all $\theta' \in R$, $\tilde{\sigma}(t|\theta) = 1$ and $\tilde{\sigma} = \sigma$ otherwise. This strategy constitutes a solution to the max-min problem.

A.3 Proof of Proposition 1

Suppose the dominant test is A-types unreliable. Suppose that only t is offered in the optimal menu and that the DM's strategy is such that $\alpha(x_1, t) = 1$ and $\alpha(x_0, t) = 0$. (Because there is only one test offered, we can restrict attention to pure strategies without loss of optimality.)

By condition (U-2), there is $\underline{\theta}$ and $\theta \in A$ with $\theta \leq_t \underline{\theta}$ such that

$$\sum_{\theta \in A, \theta \leq_t \underline{\theta}} \mu(\theta)\big(\pi_t(x_1|\underline{\theta}) - \pi_t(x_1|\theta)\big) \geq \sum_{\theta \in R, \theta \leq_t \underline{\theta}} \mu(\theta)\big(\pi_t(x_1|\underline{\theta}) - \pi_t(x_1|\theta)\big).$$

The DM can modify his strategy α at any other test t' with $\alpha(x, t') = \pi_t(x_1|\underline{\theta})$ for $x = x_1, x_0$. Then, $p_t(\alpha; \theta) \leq p_{t'}(\alpha; \theta)$ only if $\theta \leq_t \underline{\theta}$. By condition (U-2), there is at least one A-type willing to deviate. The payoffs of types choosing test t are left unchanged.

Condition (U-2) also guarantees that the DM's payoffs weakly increase for types choosing t'. Therefore it is a profitable deviation or another optimum in the max-min problem.[21]

A similar reasoning holds by applying (U-3) if $\alpha(x_1, t) = 0$ and $\alpha(x_0, t) = 1$.

Finally, if $\alpha(x_1, t) = \alpha(x_0, t) \in \{0, 1\}$, then we can always add another test with the same strategy and have at least one A-type choosing it.

Suppose the dominant test is not A-types unreliable.

I first show that whenever for any tests t, t' with $t \succeq t'$, $p_t(\alpha; \theta) - p_{t'}(\alpha; \theta)$ is single-crossing in θ with respect to the order $\geq_t$, for any α. Because $t \succeq t'$, there is $\beta : X \times X \to [0, 1]$ such that $\pi_{t'}(\tilde{x}|\theta) = \beta(x_1, \tilde{x})\pi_t(x_1|\theta) + \beta(x_0, \tilde{x})\pi_t(x_0|\theta)$ and $\sum_x \beta(\tilde{x}, x) = 1$ for $\tilde{x} = x_1, x_0$. Type $\theta \in \Theta$ prefers test t' over t if

$$\begin{aligned}&\alpha(x_1, t')\big(\beta(x_1, x_1)\pi_t(x_1|\theta) + \beta(x_0, x_1)\pi_t(x_0|\theta)\big)\\ +&\alpha(x_0, t')\big(\beta(x_1, x_0)\pi_t(x_1|\theta) + \beta(x_0, x_0)\pi_t(x_0|\theta)\big) - \alpha(x_1, t)\pi_t(x_1|\theta) - \alpha(x_0, t)\pi_t(x_0|\theta) \geq 0\end{aligned}$$

Using that $\pi_t(x_0|\theta) = 1 - \pi_t(x_1|\theta)$, the expression on the LHS is linear in $\pi_t(x_1|\theta)$. Therefore, either $p_t(\alpha; \theta) - p_{t'}(\alpha; \theta) \geq 0$ implies $p_t(\alpha; \theta') - p_{t'}(\alpha; \theta') \geq 0$ for all $\theta' \geq_t \theta$ or $p_t(\alpha; \theta) - p_{t'}(\alpha; \theta) \leq 0$ implies $p_t(\alpha; \theta') - p_{t'}(\alpha; \theta') \leq 0$ for all $\theta' \geq_t \theta$.

Suppose that condition (U-1) is not satisfied.

Suppose there is a menu with both t, t'. Take $A_1, A_2 \in A$ with $A_1 <_t A_2$ and because the signal's labels are arbitrary, suppose without loss of generality that A_1 chooses t' and A_2 chooses t in some equilibrium. Let α denote the DM equilibrium strategy in this equilibrium.

Again, because $t \succeq t'$, there is $\beta : X \times X \to [0, 1]$ such that $\pi_{t'}(\tilde{x}|\theta) = \beta(x_1, \tilde{x})\pi_t(x_1|\theta) + \beta(x_0, \tilde{x})\pi_t(x_0|\theta)$ and $\sum_x \beta(\tilde{x}, x) = 1$ for $\tilde{x} = x_1, x_0$. Type $\theta \in \Theta$ prefers test t' over t if

$$\begin{aligned}&\alpha(x_1, t')\big(\beta(x_1, x_1)\pi_t(x_1|\theta) + \beta(x_0, x_1)\pi_t(x_0|\theta)\big)\\ +&\alpha(x_0, t')\big(\beta(x_1, x_0)\pi_t(x_1|\theta) + \beta(x_0, x_0)\pi_t(x_0|\theta)\big) - \alpha(x_1, t)\pi_t(x_1|\theta) - \alpha(x_0, t)\pi_t(x_0|\theta) \geq 0.\end{aligned}$$

[21] One could worry that we do not need (U-2) and (U-3) to hold simultaneously to show that including a dominated test is optimal. But we cannot have $\alpha(x_1, t) = 1$ and $\alpha(x_0, t) = 0$ as a best-reply to t if (U-3) does not hold.

Assuming $\pi_t(x_0|\theta) > 0$, this expression becomes

$$\alpha(x_1,t')\big(\beta(x_1,x_1)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_1)\big) + \alpha(x_0,t')\big(\beta(x_1,x_0)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_0)\big) \\ - \alpha(x_1,t)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} - \alpha(x_0,t),$$

which is linear in $\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)}$, a strictly increasing function of θ (recall that $\theta \not=_t \theta'$ for all θ, θ'). We can only have $\pi_t(x_0|\theta) = 0$ if $\theta = \max \Theta$ where the $\max$ is taken with respect to $\geq_t$, so the case $\pi_t(x_0|\theta) = 0$ can be true for at most one type.

To have A_1 choose t' and A_2 choose t, the payoff difference between t' and t must be strictly decreasing in θ,[22] i.e.,

$$\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t) < 0. \tag{3}$$

A necessary condition for (3) to hold is that $\alpha(x_1,t) > 0$. Note the strict monotonicity also implies that there is $\overline{\theta} \in A$ such that any $\theta >_t \overline{\theta}$ prefers t and any $\theta \leq_t \overline{\theta}$ prefers t'. Let $A^+ = \{\theta \in A : \theta >_t \overline{\theta}\}$ and $R^+ = \{\theta \in R : \theta >_t \theta', \text{ for all } \theta' \in A\}$. Condition (U-1) not satisfied implies that only types in $A^+ \cup R^+$ choose t. Further, for any $\theta \in A^+$, $\theta' \in R^+$, $\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} < \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')}$, i.e., we have MLRP with x_0 being the high signal on $A^+ \cup R^+$. This observation combined with $\alpha(x_1,t) > 0$ imply that $\alpha(x_0,t) = 1$ (Milgrom, 1981).

But then no type ever prefer t' over t. Indeed, the condition to prefer t' over t,

$$\big(\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t)\big)\pi_t(x_1|\theta) \\ \geq \big(1 - \alpha(x_1,t')\beta(x_0,x_1) - \alpha(x_0,t')\beta(x_0,x_0)\big)\pi_t(x_0|\theta)$$

is never satisfied as the LHS is strictly negative because (3) must hold and the RHS is positive because $\beta(x_0,x_1) + \beta(x_0,x_0) = 1$ and $\alpha(\tilde{x},t') \leq 1$, $\tilde{x} = x_1, x_0$.

Thus there cannot be an equilibrium where another test than t is chosen.

Suppose that condition (U-2) is not satisfied. Suppose also that a nonsingleton menu $\mathcal{M}$ is optimal. Given that any probability of acceptance $p_{t'}(\alpha;\theta)$ with $t' \preceq t$ can be replicated with the test t, we can, for the purpose of showing that a singleton menu $\{t\}$ is optimal, focus attention to $\mathcal{M} = \{t_1, ..., t_n\}$ with $\pi_{t_i}(\cdot|\theta) = \pi_t(\cdot|\theta)$ for $i = 1, ..., n$.

[22] If the expression is constant in θ, then all types have the same probability of acceptance in t and t'. So the DM can merge t and t', have the same payoffs and not change the incentives to choose other tests.

First, observe that the set of types preferring test t_i is always an interval[23] because of the single-crossing condition on $p_t(\alpha;\theta) - p_{t'}(\alpha;\theta)$. Label tests by the position of the types choosing them in the $\geq_t$ order, i.e., t_1 is chosen by types in the lowest interval, t_2 second lowest, etc. Using the single-crossing condition again, if we eliminate t_1, all types previously preferring t_1 will choose t_2.

I now show that we can focus on solutions where $\alpha(x_0, t_1) > 0$, $\alpha(x_1, t_2) > 0$ and either (a) $\alpha(x_1, t_1) > 0 \Rightarrow \alpha(x_0, t_2) = 0$ or (b) $\alpha(x_0, t_2) > 0 \Rightarrow \alpha(x_1, t_1) = 0$.

First, I show that $\alpha(x_0, t_1) > 0$. Suppose it is not the case. Then type θ chooses t_1 over t_2 if $\alpha(x_1, t_1)\pi_t(x_1|\theta) \geq \alpha(x_1, t_2)\pi_t(x_1|\theta) + \alpha(x_0, t_2)(1 - \pi_t(x_1|\theta))$.

Rearranging, we get $\big(\alpha(x_1, t_1) - \alpha(x_1, t_2) + \alpha(x_0, t_2)\big)\pi_t(x_1|\theta) \geq \alpha(x_0, t_2)$.

To have any type choosing t_1, it must be that $\alpha(x_1, t_1) - \alpha(x_1, t_2) + \alpha(x_0, t_2) \geq 0$. But then if $p_{t_1}(\alpha;\theta) \geq p_{t_2}(\alpha;\theta)$ then $p_{t_1}(\alpha;\theta') \geq p_{t_2}(\alpha;\theta')$ for all $\theta' \geq_t \theta$. Combined, these observations contradict that t_1 is chosen by lower types than the ones choosing t_2. Therefore $\alpha(x_0, t_1) > 0$. Similarly, one can show that $\alpha(x_1, t_2) > 0$.

We must also have that either (a) $\alpha(x_1, t_1) > 0 \Rightarrow \alpha(x_0, t_2) = 0$ or (b) $\alpha(x_0, t_2) > 0 \Rightarrow \alpha(x_1, t_1) = 0$. If none of these alternatives hold, the DM must be mixing after both signals in at least one of the two tests or the DM follows the same strategy in both tests. In that case, we can eliminate t_1 with no effects on incentives or DM's payoffs.

Let $\underline{\theta}$ be the largest type with respect to $\geq_t$ choosing t_1 with positive probability. Denote by $\overline{A} = A \cap \{\theta \leq_t \underline{\theta}\}$ and $\overline{R} = R \cap \{\theta \leq_t \underline{\theta}\}$. Denote by $\tilde{\mu}(\theta) = \sigma(t_1|\theta)\mu(\theta)$. Note that by Theorem 1, we can restrict attention to A-types playing a pure strategy and therefore $\mu(\theta) = \tilde{\mu}(\theta)$ for all $\theta \in \overline{A}$.

First assume that $\alpha(x_1, t_1) > 0$ and $\alpha(x_0, t_2) = 0$. It is strictly profitable to include test t_1 in the menu if the following system of inequalities holds

$$\begin{aligned}
\alpha(x_1,t_1)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)\big) + \alpha(x_0,t_1)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_0|\theta) - \sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_0|\theta)\big)& \\
> \alpha(x_1,t_2)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)\big),& \\
\alpha(x_1,t_1)\pi_t(x_1|\theta) + \alpha(x_0,t_1)\pi_t(x_0|\theta) \geq \alpha(x_1,t_2)\pi_t(x_1|\theta) \text{ for all } \theta \leq_t \underline{\theta}.&
\end{aligned}$$

The first line guarantees that payoffs from including t_1 are larger than eliminating it and have all types choosing t_1 choosing t_2. The second line insures that types have indeed an

[23] A set Θ' is an interval if there are θ, θ' such that $\Theta' = \{\theta'' : \theta \leq_t \theta'' \leq_t \theta'\}$.

incentive to choose t_1. Note that we must have $1 \geq \alpha(x_1, t_2) > \alpha(x_1, t_1)$ for otherwise no type would pick t_2. Setting $\alpha_1 = \alpha(x_1, t_2) - \alpha(x_1, t_1)$ and $\alpha_0 = \alpha(x_0, t_1)$, we obtain the following other system of inequalities: there is $\alpha_1, \alpha_0 \geq 0$ such that

$$\alpha_1 \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_1|\theta) \big) - \alpha_0 \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_0|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_0|\theta) \big) < 0,$$
$$\alpha_1 \pi_t(x_1|\theta) - \alpha_0 \pi_t(x_0|\theta) \leq 0 \text{ for all } \theta \leq_t \underline{\theta},$$
$$\alpha_0 \leq 1,\ \alpha_1 \leq 1 - \alpha(x_1, t_1).$$

We can apply a theorem of the alternative (e.g., Rockafellar, 2015; Ball, 2023) to say that this system of inequalities cannot hold while the following does as well: there is $y, z, \lambda \geq 0$ such that

$$y \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_1|\theta) \big) + \sum_{\theta \leq_t \underline{\theta}} z_\theta \pi_t(x_1|\theta) + \lambda_1 \geq 0,$$
$$- y \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_0|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_0|\theta) \big) - \sum_{\theta \leq_t \underline{\theta}} z_\theta \pi_t(x_0|\theta) + \lambda_0 \geq 0,$$
$$(1 - \alpha(x_1, t_1)) \lambda_1 + \lambda_0 \leq 0,$$
$$(1 - \alpha(x_1, t_1)) \lambda_1 + \lambda_0 < 0 \text{ or } y > 0.$$

I will now show that a solution to the second alternative exists. First, set $\lambda = 0$.

We can rearrange the first two inequalities as

$$y \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_1|\theta) \big) \geq - \sum_{\theta \leq_t \underline{\theta}} z_\theta \pi_t(x_1|\theta) \tag{4}$$

$$y \big(\sum_{\theta \in \overline{A}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \pi_t(x_1|\theta) \big) \geq y \left(\sum_{\theta \in \overline{A}} \mu(\theta) - \sum_{\theta \in \overline{R}} \tilde{\mu}(\theta) \right) + \sum_{\theta \leq_t \underline{\theta}} z_\theta (1 - \pi_t(x_1|\theta)) \tag{5}$$

Recall that Condition (U-2) not being respected implies that

$$\pi_t(x_1|\underline{\theta}) \big(\sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta) \big) < \sum_{\theta \in A,\, \theta \leq_t \underline{\theta}} \mu(\theta) \pi_t(x_1|\theta) - \sum_{\theta \in R,\, \theta \leq_t \underline{\theta}} \mu(\theta) \pi_t(x_1|\theta). \tag{6}$$

Given that at most the type $\underline{\theta}$ is indifferent in equilibrium, if equation (6) holds for μ, it also holds by replacing μ by $\tilde{\mu}$. (Otherwise all types are indifferent between t_1 and t_2.)

Suppose first that $\sum_{\theta\in\overline{A}}\mu(\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\leq 0$. Then we can set $y=1$, $z_{\underline{\theta}}=-\sum_{\theta\in\overline{A}}\mu(\theta)+\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)$ and $z_\theta=0$ for $\theta\neq\underline{\theta}$. Both inequalities (4) and (5) become

$$\pi_t(x_1|\underline{\theta})\big(\sum_{\theta\in A,\,\theta\leq_t\underline{\theta}}\mu(\theta)-\sum_{\theta\in R,\,\theta\leq_t\underline{\theta}}\mu(\theta)\big)\leq\sum_{\theta\in A,\,\theta\leq_t\underline{\theta}}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta\in R,\,\theta\leq_t\underline{\theta}}\mu(\theta)\pi_t(x_1|\theta)$$

This inequality holds as equation (6) holds. Therefore it is profitable to eliminate t_1.

If $\sum_{\theta\in\overline{A}}\mu(\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)>0$, we get from equation (6) that $\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)>0$. In this case, we must have $\alpha(x_1,t_1)=1$ in equilibrium. Because some types choose t_2 in equilibrium, it means that $\alpha(x_0,t_1)=0$ and $\alpha(x_1,t_2)=1$. But then the DM follows the same strategy in both tests and we can eliminate one.

If instead we have $\alpha(x_0,t_2)>0$ and $\alpha(x_1,t_1)=0$. It is strictly profitable to include test t_1 in the menu if the following system of inequalities holds

$$\begin{aligned}\alpha(x_0,t_1)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_0|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_0|\theta)\big)>\alpha(x_1,t_2)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)\big)\\+\alpha(x_0,t_2)\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_0|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_0|\theta)\big),\end{aligned}$$

$$\alpha(x_0,t_1)\pi_t(x_0|\theta)\geq\alpha(x_1,t_2)\pi_t(x_1|\theta)+\alpha(x_0,t_2)\pi_t(x_0|\theta)\ \text{ for all }\theta\leq_t\underline{\theta}.$$

To have any type choosing t_1, we must have $\alpha(x_0,t_1)\geq\alpha(x_0,t_2)$. Therefore, as before, we can rewrite this system of inequality as, there is $\alpha_1,\alpha_0\geq 0$ such that

$$\alpha_1\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)\big)-\alpha_0\big(\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_0|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_0|\theta)\big)<0,$$

$$\alpha_1\pi_t(x_1|\theta)-\alpha_0\pi_t(x_0|\theta)\leq 0\ \text{ for all }\theta\leq_t\underline{\theta},$$

$$\alpha_0\leq 1-\alpha(x_0,t_2),\alpha_1\leq 1.$$

This is exactly the same system as before except that the bounds on α_1 and α_0 have changed.

If $\sum_{\theta\in\overline{A}}\mu(\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\leq 0$, we can use exactly the same solution to the alternative to show that there is no solution to the system of inequalities above. If $\sum_{\theta\in\overline{A}}\mu(\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)>0$ then by (U-2), we have $\sum_{\theta\in\overline{A}}\mu(\theta)\pi_t(x_1|\theta)-\sum_{\theta\in\overline{R}}\tilde{\mu}(\theta)\pi_t(x_1|\theta)>0$. In this case, we must have $\alpha(x_1,t_1)=1$ in equilibrium. A contradiction.

Suppose that condition (U-3) is not satisfied. This case is treated in the same way as if condition (U-2) is not satisfied by inverting the roles of x_1 and x_0.

A.4 Proof of Proposition 2

Note that in an MLRP environment, the strategy of the DM takes the form of a cutoff strategy. For each test t, there is $x_t \in X$ such that $\alpha(x,t) = 0$ for $x < x_t$, $\alpha(x,t) = 1$ for $x > x_t$ and $\alpha(x_t,t) \in [0,1]$. From Lemma 2, we know that there is an optimal menu containing the Blackwell most informative test. Because all tests are MLRP and the DM's payoffs satisfy single-crossing condition, the Lehmann order is well-defined and the Blackwell order implies the Lehmann order (Lehmann, 1988; Persico, 2000). Let $\succeq^a$ denote the Lehmann order.

The Lehmann order is defined on continuous information structure. But as outlined in Lehmann (1988), we can always make our conditional probabilities continuous by adding independent uniform between each signal. Let's assume, without loss of generality, that $X = \{1, ..., n\}$. The new distribution over signal is $\tilde{y}|\theta = \tilde{x}|\theta - u$ where u ~U$[0,1]$. Denote by F_t the cdf associated with the new information structure.

We have that $t \succeq^a t'$ if $y^*(\theta, y) \equiv F_t(y^*|\theta) = F_{t'}(y|\theta)$ is nondecreasing in θ for all y (Lehmann, 1988). In particular, this condition implies that if $F_t(y|\theta') \leq (<) F_{t'}(y'|\theta')$ then $F_t(y|\theta) \leq (<) F_{t'}(y'|\theta)$ for all $\theta > \theta'$.

Let α be the optimal strategy and x_t be the cutoff signal associated to each test. To each $\big(\alpha(\cdot,t), x_t\big)$ we can associate a $y_t = x_t - \alpha(x_t,t)$.

If t is part of an optimal menu, it must be that there is some $\theta' \in R$ such that $p_t(\alpha;\theta') \geq p_{t'}(\alpha;\theta')$ for all t'. Or put differently, $F_t(y_t|\theta') \leq F_{t'}(y_{t'}|\theta')$ for all t'. But then $F_t(y_t|\theta) \leq F_{t'}(y_{t'}|\theta)$ for all t' and all $\theta > \theta'$, in particular all $\theta \in A$. Therefore all type in A prefer test t as well and there is an solution of the max-min problem where all types in $\theta \in A$ choose t. (If there is an A-type that is indifferent between t and t' then all types in R must be indifferent or prefer t' so choosing t is an equilibrium strategy for such A-types.)

A.5 Proof of Proposition 3

Suppose that t is the only test used in the optimal menu. Define a test t' such that for all $\theta \in \Theta$,

$$\pi_{t'}(x|\theta) = \sum_{\tilde{x}=x,x'} \pi_t(\tilde{x}|\theta),$$

$$\pi_{t'}(\tilde{x}|\theta) = \pi_t(\tilde{x}|\theta), \text{ for all } \tilde{x} \neq x, x'.$$

The test t' pools signals x and x' together and is otherwise identical to t. We have $t \succ t'$ as any strategy under t' can be replicated under t.

Let μ be such that for all $\theta \neq A_1, A_2, R_1$, $\mu(\theta) \approx 0$ and such that when only t is chosen, the DM's best-reply is $\alpha(x,t) = 1$ and $\alpha(x',t) = 0$. In the max-min problem, consider the deviation $\tilde{\alpha}$ such that $\tilde{\alpha}(x,t') = \frac{\pi_t(x|A_1)}{\pi_t(x|A_1)+\pi_t(x'|A_1)} + \epsilon$ for some small $\epsilon > 0$ and $\tilde{\alpha} = \alpha$ otherwise.

Under this alternative strategy, $p_t(\alpha; A_1) < p_{t'}(\tilde{\alpha}; A_1)$ but for ϵ small enough $p_t(\alpha;\theta) > p_{t'}(\tilde{\alpha};\theta)$ for $\theta = R_1, A_2$. Therefore A_1 is accepted with strictly higher probability and the other types with the same probability as they choose the same test. If the prior on other types is sufficiently small, the deviation is still strictly profitable for the DM.

A.6 Proof of Proposition 4

($\Rightarrow$) The proof is similar to the one in Milgrom (1981). Denote by $G_t(\cdot|x)$ the cdf of posterior beliefs after signal x in test t. For all $\theta > \theta'$,

$$\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Take some $\theta^* \geq \theta'$. Summing over θ, we get

$$\sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \sum_{\theta>\theta^*}\mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Inverting and summing over θ', we get

$$\frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_t(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_t(x|\theta)} \leq \frac{\sum_{\theta^*\geq\theta'}\mu(\theta')\pi_{t'}(x|\theta')}{\sum_{\theta>\theta^*}\mu(\theta)\pi_{t'}(x|\theta)}$$

which implies

$$\frac{G_t(\theta^*|x)}{1-G_t(\theta^*|x)} \leq \frac{G_{t'}(\theta^*|x)}{1-G_{t'}(\theta^*|x)} \quad \Rightarrow \quad G_t(\theta^*|x) \leq G_{t'}(\theta^*|x)$$

($\Leftarrow$) Take $\theta > \theta'$ and let $\mu(\theta) + \mu(\theta') = 1$ (i.e., there is probability zero on other types). If $\mu(\cdot|t,x) \succeq_{FOSD} \mu(\cdot|t',x)$, then we have $\mu(\theta|t,x) \geq \mu(\theta|t',x)$ and $\mu(\theta'|t,x) \leq \mu(\theta'|t',x)$. By Bayes' rule these inequalities imply $\frac{\pi_t(x|\theta)}{\pi_{t'}(x|\theta)} \geq \frac{\sum_{\theta''}\mu(\theta'')\pi_t(x|\theta'')}{\sum_{\theta''}\mu(\theta'')\pi_{t'}(x|\theta'')}$ and $\frac{\pi_t(x|\theta')}{\pi_{t'}(x|\theta')} \leq \frac{\sum_{\theta''}\mu(\theta'')\pi_t(x|\theta'')}{\sum_{\theta''}\mu(\theta'')\pi_{t'}(x|\theta'')}$. Rearranging, we obtain $\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$.

A.7 Proof of Proposition 5

The way this proof proceeds is by fixing a menu and dividing tests in two categories: (1) those for which $\alpha(x_0, \tilde{t}) \in (0,1)$ and $\alpha(x_1, \tilde{t}) = 1$ and (2) $\alpha(x_0, \tilde{t}) = 0$ and $\alpha(x_1, \tilde{t}) \in (0,1]$. I exclude the possibility that the DM always accepts or rejects after any signal as it would either be the only test chosen in equilibrium or never chosen. Then, I show that within each category, it is without loss of optimality to have at most one test. It is thus optimal to have at most two tests in the menu. The last part of the proof shows that the resulting menu is dominated by having only one test.

If there are two tests, $t > t'$ such that $\alpha(x_0, \tilde{t}) = 0$ and $\alpha(x_1, \tilde{t}) \in (0,1]$, I will show that, $p_t(\alpha; \theta') \geq p_{t'}(\alpha; \theta') \Rightarrow p_t(\alpha; \theta) \geq p_{t'}(\alpha; \theta)$ for all $\theta > \theta'$. Take two tests such that $\alpha(x_0, \tilde{t}) = 0$, $t > t'$. Let α, α' denote their respective probability of accepting after x_1. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_1|\theta) = 0$. Rearranging, $\alpha(\theta) = \alpha' \frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. From our assumption on the difficulty environment, $\alpha(\theta)$ is decreasing in θ. If $p_t(\alpha; \theta') \geq p_{t'}(\alpha; \theta')$ for some θ' then $\alpha \geq \alpha(\theta')$. Then $\alpha \geq \alpha(\theta)$ for all $\theta > \theta'$.

In equilibrium, we must have that there is one $\theta' \in R$ that chooses t and thus for all $\theta \in A$, $p_t(\alpha; \theta) \geq p_{t'}(\alpha; \theta)$. Then there is an solution of the max-min problem where t' is never chosen. A similar argument can be made for all tests where $\alpha(x_0, \tilde{t}) > 0$.

Thus we conclude that it is without loss of optimality that the optimal menu has at most two tests.

Suppose the optimal menu uses two tests, $t > t'$. I will now show that it must be that $\alpha(x_0, t) \in (0,1)$ and $\alpha(x_1, t') \in (0,1)$, i.e., the DM must accept in the hard test when there is a fail grade and accept in the easy test only if there is a pass grade. Suppose it is not the case and denote by α, α' their respective mixing probabilities. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_0|\theta) - \pi_{t'}(x_1|\theta) = 0$, which is equivalent to $\alpha(\theta) = \alpha' \frac{1}{\pi_t(x_1|\theta)} + (1 - \alpha') \frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. Again from our assumptions, this function is decreasing in θ. A type θ chooses t if $\alpha \geq \alpha(\theta)$. Thus if one $\theta \in A$ chooses t all $\theta \in R$ choose t and there is no pooling of A and R-types on t', or it is payoff equivalent to just offering t. Therefore, $\alpha(x_0, t) \in (0,1)$ and $\alpha(x_1, t') \in (0,1)$ for $t > t'$. If the DM mixes, he must be indifferent and thus we have

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta)\pi_t(x_0|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')\pi_t(x_0|\theta') = 0$$
$$\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') = 0$$

In the easy test, because the DM rejects with positive probability after x_1 and rejects for sure after x_0 (as he uses a cutoff strategy), his payoffs from t' is 0, i.e., he does as well as rejecting for sure.

In the hard test, he accepts with some probability after x_0 and thus his payoffs are

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta'),$$

that is the payoffs he would get from accepting all types choosing t. Thus the overall payoffs from the menu is $\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$. Offering a menu is better than a singleton menu if this value is strictly greater than offering t and following the signal

$$\begin{aligned}\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta') &> \sum_{\theta \in A} \mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\pi_t(x_1|\theta') \\ &= \sum_{\theta \in A} \sigma(t|\theta)\mu(\theta)\pi_t(x_1|\theta) + \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) \\ &- \sum_{\theta' \in R} \sigma(t|\theta')\mu(\theta')\pi_t(x_1|\theta') - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

We can rearrange and use the indifference condition at (x_0, t) to get

$$0 > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta').$$

This inequality implies

$$0 > \frac{\sum_{\theta \in \Theta} \sigma(t'|\theta)\mu(\theta)\pi_{t'}(x_1|\theta)}{\sum_{\theta \in \Theta} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta)} \left(\sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta') \right).$$

Using the indifference condition at (x_1, t'), we can replace 0 on the LHS and get

$$\begin{aligned}&\frac{1}{\sum_{\theta} \sigma(t'|\theta)\mu(\theta)\pi_{t'}(x_1|\theta)} \left(\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') \right) \\ &\quad > \frac{1}{\sum_{\theta} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta)} \left(\sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta') \right)\end{aligned}$$

But because t is more difficult than t', we have $\mu(\theta|x_1, t) \succeq_{FOSD} \mu(\theta|x_1, t')$. Thus we get a contradiction.

A.8 Proof of Proposition 6

Proof. Suppose condition (2) holds.

Suppose that all types choose the same test testing dimension j. Take $(\tilde{\theta}_i, \tilde{\theta}_j) \in \arg\min_{\theta \in A} p_{t_j}(\alpha; \theta)$. Because $p_{t_j}(\alpha; \theta_i, \theta_j)$ is constant in θ_i, we have $(\overline{\theta}_i, \tilde{\theta}_j) \in \arg\min_{\theta \in \Theta} p_{t_j}(\alpha; \theta)$ as well and from condition (2), $(\overline{\theta}_i, \tilde{\theta}_j) \in A$. Consider the deviation in the problem with commitment to $(\tilde{\alpha}, \tilde{s})$ such that for t_i,

- $\tilde{\alpha}(\cdot, t_i)$ is set so that it has a cutoff structure and $p_{t_i}(\tilde{\alpha}; \overline{\theta}_i, \tilde{\theta}_j) = p_{t_j}(\alpha; \overline{\theta}_i, \tilde{\theta}_j) + \epsilon$ and $\tilde{\alpha}(\cdot, t_j) = \alpha(\cdot, t_j)$ otherwise.
- $\tilde{\sigma}(t_i | \overline{\theta}_i, \tilde{\theta}_j) = 1$ and $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ otherwise.

Because the test t_i has the strict MLRP when restricting attention to dimension i, for all $\theta_i < \overline{\theta}_i$, $p_{t_i}(\tilde{\alpha}; \overline{\theta}_i, \theta_j) = \min_{\theta \in \Theta} p_{t_j}(\alpha; \theta) + \epsilon > p_{t_i}(\tilde{\alpha}; \theta_i, \theta_j)$. Therefore, for $\theta_i < \overline{\theta}_i$, $p_{t_j}(\alpha; \theta) > p_{t_i}(\tilde{\alpha}; \theta_i, \theta_j)$ if ϵ is small enough. These inequalities imply that no other type has an incentive to choose test i but $(\overline{\theta}_i, \tilde{\theta}_j) \in A$ is accepted with strictly higher probability. Thus the menu with only test j cannot be the optimal menu.

Suppose condition (2) does not hold.

If condition (2) is not satisfied, then there a dimension, say 1, and $\tilde{\theta}_2 \in \Theta_2$ such that $(\overline{\theta}_1, \tilde{\theta}_2) \in R$. By the monotonicity of payoffs in the bidimensional environment, we have that $(\theta_1, \tilde{\theta}_2) \in R$ for all $\theta_1 \in \Theta_1$. Moreover, for all $\theta_2 < \tilde{\theta}_2$ and all $\theta_1 \in \Theta_1$, $(\theta_1, \theta_2) \in R$.

Now suppose μ is such that $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$. And that $\mu(\theta_1, \theta_2) \approx 0$ for all $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$.

I am going to show that $\{t_2\}$ is optimal when t_1 fully reveals dimension 1. Because this test can replicate the strategies of any t_1, it is enough to prove our claim. Suppose there is an optimal menu $\{t_1, t_2\}$. From our assumptions on μ, the DM follows a cutoff strategy after t_2. That's because his payoff is monotone along that dimension, ignoring $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$ whose prior probability is close to zero. So it does not upset the cutoff structure of the best-response. The cutoff strategy implies that $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ because the likelihood ratio is strictly increasing.

Suppose that some $(\theta_1, \tilde{\theta}_2)$ chooses t_1 with probability 1 in equilibrium. Because $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$, it must be that the best-response is $\alpha(x = \theta_1, t_1) = 0$ (recall that t_1 fully reveals θ_1). Thus $p_{t_2}(\alpha; \theta_1, \theta_2) = 0$ for all $\theta_2 \in \Theta_2$, otherwise there is a profitable deviation. Either this contradicts the fact that the DM best replies or in equilibrium the DM rejects after all signals in every test. But then he is weakly better off only offering t_2.

Thus to have $\{t_1, t_2\}$ strictly better, it must be that all $(\theta_1, \tilde{\theta}_2)$ choosing t_1 mix in equilibrium. This mixing means that $p_{t_1}(\alpha; \theta_1, \tilde{\theta}_2) = p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$. But by the cutoff structure of $\alpha(\cdot, t_2)$ and the strict MLRP assumption, we have $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ and $p_{t_2}(\alpha; \theta_1, \theta_2) < p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 < \tilde{\theta}_2$. Thus t_2 is strictly preferred for all $(\theta_1, \theta_2) \in A$. Thus choosing only $\{t_2\}$ is an optimal menu. $\square$

A.9 Proof of Theorem 2

The proof follows closely the one of Theorem 1. Let $T' = \{(t, c) : t \in T, c \in C\}$. To simplify the notation, I will study the problem using T' as the set of feasible test. The problem the DM needs to solve when committing to α is

$$\max_{\alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T', \tilde{\theta} \in e(\theta, t)} \tilde{\sigma}(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta}) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T', \tilde{\theta} \in e(\theta', t)} \tilde{\sigma}(t, \tilde{\theta}|\theta') p_t(\alpha; \tilde{\theta})$$
$$\text{s.t. } \tilde{\sigma}(\cdot|\theta) \in \arg\max \sum_{t \in T', \tilde{\theta} \in e(\theta, t)} \sigma(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta}), \quad \text{for all } \theta \in \Theta$$

Therefore, $\sum_{t,\tilde{\theta}} \tilde{\sigma}(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta}) = \max_{\sigma(\cdot|\theta)} \sum_{t,\tilde{\theta}} \sigma(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta})$ for all θ. We can plug this expression in the DM's maximisation problem to obtain

$$\max_{\alpha} \max_{\sigma^A} \min_{\sigma^R} \sum_{\theta \in A} \mu(\theta) \sum_{t,\tilde{\theta}} \sigma(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta}) - \sum_{\theta' \in R} \mu(\theta') \sum_{t,\tilde{\theta}} \sigma(t, \tilde{\theta}|\theta') p_t(\alpha; \tilde{\theta})$$

where the $\min$ is obtained because of the minus sign.

Let $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ and $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Note that σ^A is fixed in the min-max problem. I will now show that these strategies are equilibrium strategies.

Because the order of maximisation does not matter, $\sigma^A \in \arg\max \min v(\alpha, \tilde{\sigma}^A, \tilde{\sigma}^R)$. Moreover, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R) \Leftrightarrow \sum_{t,\tilde{\theta}} \sigma(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta}) \geq \sum_{t,\tilde{\theta}} \tilde{\sigma}(t, \tilde{\theta}|\theta) p_t(\alpha; \tilde{\theta})$ for all $\theta \in A$ and $\tilde{\sigma}^A$. This last expression does not depend σ^R. Therefore, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R)$.

Similarly, $\alpha \in \arg\max \min v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Because v is linear in both α and σ^R and that the set of strategies are convex, (α, σ^R) is a saddle-point of $v(\cdot, \sigma^A, \cdot)$ by the minimax theorem. As for σ^A, $\sigma^R \in \arg\min v(\alpha, \sigma^A, \tilde{\sigma}^R) \Leftrightarrow \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) \geq \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha; \theta)$ for all $\theta \in R$ and $\tilde{\sigma}^R$. Therefore all strategies are best-reply.

Beliefs on-path are formed using Bayes' rule and off-path beliefs are chosen to justify the off-path actions of α (this is always possible to find as any action is best-reply to some beliefs).

Note also that we can without loss of generality take σ^A to be in pure strategy as it is maximises a linear function.

The only thing we need prove is that it is optimal to have a different message for each type $\theta \in A$, the rest follows from Theorem 1. Suppose it is not the case and take a solution (α, σ^A) of the max-min problem where the A-types play a pure strategy.

Then, there is $\theta_1, \theta_2 \in A$ and $(t, c) \in T \times C$ such that $\sigma(t, c, \tilde{\theta}_1|\theta_1) = \sigma(t, c, \tilde{\theta}_1|\theta_2) = 1$ (if they use a different test then we can also change the message and nothing is changed). Consider the alternative strategy α' where, for some unused (t, c') in the original mechanism, $\alpha'(t, c', x) = \alpha(t, c, x)$ for all $x \in X$ and $\alpha'(t'', c'', x) = \alpha'(t'', c'', x)$ for all other $(t'', c'') \in T \times C$ and all $x \in X$ otherwise. The new strategy α' is thus the same as α but makes sure that if the pair (t, c') is chosen, it uses the same actions as (t, c). Now consider the following strategy $\tilde{\sigma}^A$ in the auxiliary max-min problem, $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ for $\theta \neq \theta_1$ and $\tilde{\sigma}(t, c'|\theta_1) = 1$. This strategy gives the same value in the max-min problem as under (α, σ^A). Moreover, any deviations under α' gives the same payoff than under α. Therefore, $(\alpha', \tilde{\sigma}^A)$ is an solution to problem with commitment.

A.10 Proof of Proposition 7

Suppose $e(t, \theta) = \{\theta' : \theta' \leq_t \theta\}$. Denote by $\underline{\theta}_t$ a type such that $\underline{\theta}_t \leq_t \theta$ for all θ. Take an equilibrium with productive communication and only one test is used. I will show that this equilibrium must be payoff equivalent to a setting where all types send the same message and choose different tests. If $\alpha(L, t, c) = \alpha(H, t, c) \in \{0, 1\}$, this strategy would give either a strict incentive to all types to choose message c or a strict incentive not to choose it. Suppose it is not the case. After any message c and test t, there are four possibilities.

(1) if $\alpha(L, t, c) > \alpha(H, t, c)$, all types sending message c have a strict incentive to copy $\underline{\theta}_t$ and this strategy is equilibrium behaviour only if $\sum_{\theta \in A} \mu(\theta)\sigma(t, c, \underline{\theta}_t|\theta) = \sum_{\theta \in R} \mu(\theta)\sigma(t, c, \underline{\theta}_t|\theta)$.

(2) if $0 < \alpha(L, t, c) = \alpha(H, t, c) < 1$, it must hold that $\sum_{\theta \in A} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta) = \sum_{\theta \in R} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta)$.

(3) if $0 < \alpha(L, t, c) < \alpha(H, t, c) < 1$, then it is a strictly dominant strategy to copy itself for any type. It also has to be that $\sum_{\theta \in A} \mu(\theta)\sigma(t, c, \theta|\theta)\pi_t(x|\theta) = \sum_{\theta \in R} \mu(\theta)\sigma(t, c, \theta|\theta)\pi_t(x|\theta)$ for $x = L, H$.

(4) if $0 \leq \alpha(L, t, c) < \alpha(H, t, c) \leq 1$, with at least one equality, then there are two

possibilities:

(a) $0 = \alpha(L, t, c) < \alpha(H, t, c) \leq 1$

(b) $0 < \alpha(L, t, c) < \alpha(H, t, c) = 1$.

Note that there cannot be two messages c, c' that are in case (4) as one would message would strictly dominate the other for any type.

In both case (1) and (2), the acceptance probability is the same for each type. So if we have two messages in case (1) and (2), respectively, say c_1, c_2, either one message dominates the other or we can construct another equilibrium where all types that sent c_1 now send c_2 and copy $\underline{\theta}_t$. This new strategy would not change any player's payoffs or incentives because we have added an equal mass of A and R-types.

Let c_3, c_4 be two messages on path such that we are in case (3) and (4). Note that it must always be that if $p_{t,c_4}(\alpha; \theta) \geq p_{t,c_3}(\alpha; \theta)$, then $p_{t,c_4}(\alpha; \theta') > p_{t,c_3}(\alpha; \theta')$ for all $\theta' \geq_t \theta$.

Suppose there is a solution (α, σ) to the max-min problem where only test t is chosen and only one message in case (1) or (2) and one message in case (4). Let θ^* be the largest type sending a message in case (3) and c^* a message in case (3) θ^* is sending. Let c_4 the message in case (4).

There is another solution to the max-min problem (α', σ') where the DM picks an arbitrary test t' and sets $\alpha'(L, t', c) = \alpha'(H, t', c) = p_{t,c^*}(\alpha; \theta^*)$ and $\alpha'(x, t, c) = \alpha(x, t, c_4)$. All types $\theta \geq_t \theta^*$ choose $\sigma'(t, c|\theta) = \sigma(t, c_4|\theta)$. All types $\theta \leq_t \theta^*$ choose $\sigma'(t', c|\theta) = \sum_{c' \neq c_4} \sigma(t, c'|\theta)$.

For all message $c' \neq c_4$, we have $\sum_{\theta \in A} \mu(\theta)\sigma(t, c', \theta|\theta) = \sum_{\theta \in R} \mu(\theta)\sigma(t, c', \theta|\theta)$, thus, we also have $\sum_{c' \neq c_4} \sum_{\theta \in A} \mu(\theta)\sigma(t, c', \theta|\theta) = \sum_{c' \neq c_4} \sum_{\theta \in R} \mu(\theta)\sigma(t, c', \theta|\theta)$. Therefore, setting a constant acceptance probability after (t', c) gives the same payoff to the DM than under the original solution. For all types $\theta \leq_t \theta^*$ choosing (t', c) gives weakly higher payoffs and σ' is therefore also a solution to the max-min problem.

The payoffs from (t, c_4) under the original solution to the max-min problem are the same as the payoffs from (t, c) under the new solution for all players. Therefore, we have constructed a new solution to the max-min problem where only one message is sent. The example in Section 3 shows that there are cases where the DM can strictly benefit from a menu.

Suppose that $e(t, \theta) = \{\theta\}$ *for all* $t \in T$. The result follows from Lemma 2. If a Blackwell dominated test is chosen by an A-type, then we can introduce a message test pair (c, t) in the max-min problem that will make the A-type better off without making any R-type better off. This deviation will create a new solution to max-min problem.

B Mechanism and randomisation

So far we have restricted attention to menus of tests. But the DM could potentially use a more elaborate *mechanism* to allocate tests to agents, possibly randomly. I define a mechanism $\tau : \Theta \to \Delta T$, a possibly random mapping from type report to distribution over tests. A strategy for the DM remains a mapping from test allocation and signal realisation to an acceptance decision, $\alpha : T \times X \to [0, 1]$, and the agent's strategy is a mapping from type to type report. I assume that the DM cannot observe the type report but we can naturally extend the mechanism τ to allow for output messages in the spirit of Section 5. Standard revelation principle arguments show that it is without loss of generality to restrict attention to type reports.

The DM's problem is now to maximise his expected payoff subject to incentive-compatibility constraints. In the baseline model, the DM cannot commit to its strategy α. Let $BR(\tau) := \{\alpha : \mathbb{E}_{\alpha,\tau}[v(a, \theta)] \geq \mathbb{E}_{\alpha',\tau}[v(a, \theta)]$, for all $\alpha'\}$, the set of best-replies when the mechanism is τ. The DM's problem is

$$\begin{aligned}
&\max_{\tau,\alpha} \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) \\
&\text{s.t.} \sum_t (\tau(t|\theta) - \tau(t|\theta')) p_t(\alpha; \theta) \geq 0 \text{ for all } \theta, \theta' \\
&\qquad \alpha \in BR(\tau)
\end{aligned}$$

The first constraint is the incentive-compatibility constraint of type θ deviating to θ' and the second constraint ensures that the DM best replies to the information revealed by the output of the mechanism.[24]

By using a mechanism, the agent can be randomly allocated to different tests without being indifferent between them. On the other hand, when restricting attention to menus, the agent has to be indifferent between tests if he randomises over tests. In the following example (inspired by Glazer and Rubinstein, 2004), I show that access to a mechanism can strictly improve the DM's payoffs.

Example (Randomised allocation)**.** Suppose there are six types $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_i = 0, 1,\ 1 \leq \theta_1 + \theta_2 + \theta_3 \leq 2\}$ and $A = \{(\theta_1, \theta_2, \theta_3) : \theta_1 + \theta_2 + \theta_3 = 2\}$. The DM has access to three tests, each perfectly revealing one dimension: $T = \{1, 2, 3\}$ with $\pi_t(x = \theta_t|\theta) = 1$.

[24] This definition does not put constraints on off-path optimality, but because any strategy is a best-reply to some beliefs, satisfying $BR(\tau)$ will lead to a weak PBE.

The prior μ is uniform. At the optimum, the DM accepts when the signal is equal to 1. The optimal mechanism τ allocates each A-type with probability $1/2$ to each test where their dimension is equal to 1. Each R-type is allocated with probability $1/2$ to the dimension where it has value 1 and $1/4$ in the other dimensions.

This mechanism accepts A-types with probability one and accepts R-types with probability $1/2$. In particular, the mechanism randomises the allocation of R-types over tests they are not indifferent between. If the DM could use only a menu, the R-types would always choose the test that reveals their dimension equal to one. Thus any menus and strategies that accept R-types with probability $1/2$ must also accept A-types with probability $1/2$.

I now show that the optimal mechanism can also be characterised by a max-min problem and the DM does not benefit from commitment.

To set up the characterisation of the optimal mechanism, let $s : A \to \Delta T$ and $m : R \to \Delta A$, and abusing notation, let $\alpha : T \times X \to [0, 1]$ and

$$\begin{aligned} v(\alpha, s, m) &\equiv \sum_{\theta \in A} \sum_{t \in T} s(t|\theta) \Big[\mu(\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') m(\theta|\theta') p_t(\alpha; \theta')\Big] \\ &= \sum_{\theta \in A} \sum_{t \in T} \mu(\theta) s(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{\theta \in A} m(\theta|\theta') \sum_{t \in T} s(t|\theta) p_t(\alpha; \theta') \end{aligned} \tag{7}$$

The function s can be interpreted as A-types choosing a test, m as R-types choosing an A-type to mimic, and α as the DM accepting the agent after a test and signal realisation. The function v is then the DM's expected payoffs from a distribution over tests induced by the pair (s, m). I explain these objects in more detail in the discussion of Theorem 3.

Theorem 3. *The value of an optimal mechanism is*

$$V = \max_{\alpha, s} \min_{m} v(\alpha, s, m) \tag{8}$$

For any $(\alpha, s) \in \arg\max_{\tilde{\alpha}, \tilde{s}} \min_{\tilde{m}} v(\tilde{\alpha}, \tilde{s}, \tilde{m})$ *and* $m \in \arg\min_{\tilde{m}} \max_{\tilde{\alpha}} v(\tilde{\alpha}, s, \tilde{m})$*, an optimal mechanism is*

- *for* $\theta \in A : \tau(t|\theta) = s(t|\theta)$
- *for* $\theta' \in R : \tau(t|\theta') = \sum_{\theta \in A} m(\theta|\theta') \tau(t|\theta)$
- *the DM's strategy is* α

Moreover, the DM does not benefit from committing to α*.*

Theorem 3 provides another characterisation of the optimal mechanism in terms of a max-min problem. As in Theorem 1, the that commitment has no value follows from the max-min structure of the characterisation. To understand the structure of this max-min problem better, consider the objective function v for a fixed α. This can be interpreted as a zero-sum game where where the maximiser, the A-types, chooses $s : A \to \Delta T$ and the minimiser, the R-types, chooses $m : R \to \Delta A$. The payoffs of a given A-type θ choosing test t and a given R-type, θ', choosing an A-type $\tilde{\theta}$ can be expressed as:

$$\begin{aligned}
&\text{for } \theta \in A \text{ choosing } t,\ \mu(\theta)p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta')m(\theta|\theta')p_t(\alpha;\theta') \\
&\text{for } \theta' \in R \text{ choosing } \tilde{\theta},\ \mu(\theta')\sum_t s(t|\tilde{\theta})p_t(\alpha;\theta') - \sum_{\theta \in A,t} \mu(\theta)s(t|\theta)p_t(\alpha;\theta)
\end{aligned}$$

In the payoffs of the R-type, his strategy, the choice of $\tilde{\theta}$, affects only the first part of the payoffs. So the R-type is effectively trying to maximise his probability of being accepted. On the other hand, the A-types maximise a modified version of their utility where they maximise their probability of being accepted while being penalised every time a R-type mimics them and is accepted. The A-types' utility is thus modified to align it with the DM's payoffs. The induced distribution over tests determines the optimal mechanism when strategy α is used.

It is worth noting that given the max-min structure of the optimal mechanism, the DM can also benefit from allocating agents to dominated tests under the same conditions as in Section 4.1.

Proof. The designer's problem is

$$\begin{aligned}
\max_{\tau,\alpha}\ & \sum_{\theta \in A} \mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) \\
\text{s.t. } & \sum_t (\tau(t|\theta) - \tau(t|\theta'))p_t(\alpha;\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \alpha \in BR(\tau)
\end{aligned}$$

If the DM could commit over a strategy α, his problem would be

$$\begin{aligned}\tilde{V}(\alpha) = \max_{\tau} \; & \sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) \\ \text{s.t. } & \sum_t (\tau(t|\theta)-\tau(t|\theta'))p_t(\alpha;\theta) \geq 0 \text{ for all } \theta,\theta'\end{aligned}$$

Step 1: Show that $\tilde{V}(\alpha) = \max_s \min_m v(\alpha, s, m)$ where v is defined in (7).

To show this claim, I am going to relax the mechanism design problem by restricting attention to the IC constraints of R-types deviating to reporting an A-type:

$$\begin{aligned}\tilde{V}(\alpha) = \max_{\tau} \; & \sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) \\ \text{s.t. } & \sum_t \tau(t|\theta)p_t(\alpha;\theta) \geq \max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta), \;\; \text{for all } \theta\in R\end{aligned}$$

The IC constraints are written to express that reporting type θ for $\theta \in R$ is better than any other reporting strategy over the A-types.

Now note that if an IC constraint is slack at the optimum, we could improve the DM's payoff by setting $\tau(t|\theta) = \sum_{\theta'\in A} m^*(\theta'|\theta)\tau(t|\theta')$ with

$$m^* \in \arg\max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta).$$

That would reduce the probability of type $\theta \in R$ of being accepted and would not change any other constraints in the relaxed problem. Thus at the optimum,

$$\sum_t \tau(t|\theta)p_t(\alpha;\theta) \geq \max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta).$$

We can plug this expression in the payoffs to get

$$\begin{aligned}\tilde{V}(\alpha) &= \max_{\tau}\sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\max_{m(\cdot|\theta)}\sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta) \\ &= \max_{\tau}\min_{m}\sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta)\end{aligned}$$

Note that we can take out the max of the summation by the linearity of the expression in m and it becomes a min because of the minus sign. This expression also corresponds to

v as defined in (7).

It remains to show that the solution of this relaxed mechanism is indeed optimal. Take $s \in \arg\max \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ and $m \in \arg\min v(\alpha, s, \tilde{m})$ and define the optimal mechanism by

- $\tau(t|\theta) = s(t|\theta)$ for $\theta \in A$
- $\tau(t|\theta') = \sum_{\theta \in A} m(\theta|\theta') s(t|\theta)$ for $\theta' \in R$

Note that an outcome of this mechanism gives payoff weakly higher than $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ as $v(\alpha, s, m) \geq \min_{\tilde{m}} v(\alpha, s, \tilde{m})$. Thus if it is incentive-compatible, it must be actually equal to the upper bound $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$.

Note that all allocations are either allocations of A-types or convex combinations of the A-types' allocations thus it is enough to check that no type has any incentive to report any A-type in the mechanism.

By definition of m,

$$v(\alpha, s, m) \geq v(\alpha, s, m'), \text{ for all } m'$$

Therefore, for each R-type, θ',

$$\sum_{\theta \in A} m(\theta|\theta') \sum_t \tau(t|\theta) p_t(\alpha; \theta') \geq \sum_t \tau(t|\tilde{\theta}) p_t(\alpha; \theta'), \text{ for any } \tilde{\theta} \in A$$

For an A-type θ, consider the choice of choosing $\tilde{s}$ such as $\tilde{s}(\cdot|\theta) = s(\cdot|\tilde{\theta})$ for some $\tilde{\theta} \in A$ and the same otherwise. By definition of s,

$$v(\alpha, s, m) = \min_{\tilde{m}} v(\alpha, s, \tilde{m}) \geq \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$$

Rearranging,

$$\begin{aligned} \mu(\theta) \sum_t \big(s(t|\theta) - s(t|\tilde{\theta})\big) p_t(\alpha; \theta) \geq & \max_{\tilde{m}} \sum_{\theta' \in R} \mu(\theta') \sum_{\theta'' \in A} \tilde{m}(\theta''|\theta') \sum_t s(t|\theta) p_t(\alpha; \theta') \\ & - \max_{\tilde{m}} \sum_{\theta' \in R} \mu(\theta') \sum_{\theta'' \in A} \tilde{m}(\theta''|\theta') \sum_t \tilde{s}(t|\theta) p_t(\alpha; \theta') \end{aligned}$$

where the min is transformed in max because of the negative sign. Note that the LHS is the IC constrain of type θ deviating to type $\tilde{\theta}$. The RHS is the difference in the probability of the R-types of being accepted when they choose a mimicking strategy $\tilde{m}$. Note that the only difference between s and $\tilde{s}$, from their point of view, is that there is weakly less choice of allocations to mimic because we have θ choosing the same allocation as $\tilde{\theta}$. Therefore it must be that the RHS is positive, which implies that the LHS is as well.

Step 2: Show that the DM does not benefit from commitment in the optimal mechanism.

Take $(\alpha, s) \in \arg\max_{\tilde{\alpha},\tilde{s}} \min_{\tilde{m}} v(\tilde{\alpha}, \tilde{s}, \tilde{m})$ and $m \in \arg\min_{\tilde{m}} \max_{\tilde{\alpha}} v(\tilde{\alpha}, s, \tilde{m})$. The α selected would be the optimal strategy when the DM can commit.

Note that because the order of maximisation does not matter, we also have $\alpha \in \arg\max_{\tilde{\alpha}} \min_{\tilde{m}} v(\tilde{\alpha}, s, \tilde{m})$. Note that v is linear in $\tilde{\alpha}$ and $\tilde{m}$ and thus by the minimax theorem,

$$v(\alpha, s, m) \geq v(\alpha', s, m), \text{ for all } \alpha'$$
$$v(\alpha, s, m) \leq v(\alpha, s, m'), \text{ for all } m'$$

Thus α best-replies to the optimal mechanism when the DM can commit and m is also a best reply to (α, s), thus satisfying the condition for characterising the equilibrium in Step 1. □

C Additional max-min characterisation result

The goal of this section is to show how the max-min characterisation can be useful to show that commitment has no value in environments where payoffs are not necessarily linear in strategies or the strategy space is not convex. In particular, I show that we can still have a similar result as in Theorem 1 if we restrict attention to DM strategies that take a cutoff form.

I assume the following. The set of signals is $X \subset \mathbb{R}$ and any DM strategy is a member of the set of cutoff strategies $C = \{\alpha : \text{ for all } t \in T, x' > x, \alpha(x,t) > 0 \Rightarrow \alpha(x',t) = 1\}$. Note that C is a compact subset of $\mathbb{R}^{|X|} \times \mathbb{R}^{|T|}$. However, it is not convex.[25] For the next result, I will also assume that the testing technology is simple and consists of two tests, $T = \{t, t'\}$.

Proposition 8. *Suppose $X \subset \mathbb{R}$ and $T = \{t, t'\}$. Suppose also that the DM is restricted to using cutoff strategies.*

[25]This cutoff strategy can be viewed as a subset of $\mathbb{R}$ for each test but then the payoffs are not linear or concave in α.

For any $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}\in C, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$*, an optimal menu is*

$$\mathcal{M} = \cup_{\theta \in A} \operatorname{supp} \sigma(\cdot|\theta).$$

Additionally, for any $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}\in C} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$*,* $(\alpha, \sigma^A, \sigma^R)$ *are the strategies in the corresponding DM-preferred equilibrium.*

Moreover, the DM does not benefit from commitment.

Proof. The proof follows identical lines as the proof of Theorem 1. The only difference is that we cannot apply the minimax theorem in the zero-sum game between the R-types and the DM. To prove the Nash Equilibrium of this game exists, I will use techniques that rely on the monotonicity of the best-reply correspondences.

We endow the set C with the following partial order:

$$\alpha \geq_C \alpha' \Leftrightarrow \alpha(x,t) \leq \alpha'(x,t) \text{ and } \alpha(x,t') \geq \alpha'(x,t'), \text{ for all } x \in X.$$

Note that for any $\alpha, \alpha' \in C$, either $\alpha(x,t) \geq \alpha'(x,t)$ for all $x \in X$ or $\alpha(x,t) \leq \alpha'(x,t)$ for all $x \in X$, i.e., in the test t, either the cutoff to accepting is lower in α or in α'.

We endow $\Sigma^R = \{\sigma : R \to \Delta T\}$ with the following partial order

$$\sigma \geq_R \sigma' \Leftrightarrow \sigma(t|\theta) \geq \sigma'(t|\theta), \text{ for all } \theta \in R.$$

Recall that T is binary so this guarantees that $\geq_I$ is antisymmetric. It is also transitive and reflexive and is therefore a partial order.

Both $(C, \geq_C)$ and $(\Sigma^R, \geq_R)$ are lattices.

Let $BR_R(\alpha)$ and $BR_D(\sigma)$ denote the best-reply correspondences of the R-types and the DM. These correspondences are not empty because each player maximises a continuous function over a compact domain.

Abusing notation, denote by $v(\alpha, \sigma)$ with $\sigma \in \Sigma^R$, the payoffs of the DM when the strategies of the A-types is fixed. Let $v(\alpha, \sigma|t)$ be the payoffs in test t. I will now show that $v(\alpha, \sigma)$ satisfies the single-crossing property in $(\alpha; \sigma)$ (Milgrom and Shannon, 1994): for any $\sigma' \geq_R \sigma$ and $\alpha' \geq_C \alpha$, $v(\alpha', \sigma) \geq (>) v(\alpha, \sigma) \Rightarrow v(\alpha', \sigma') \geq (>) v(\alpha, \sigma')$.

Let $\delta(\theta) = \sigma'(t|\theta) - \sigma(t|\theta) \geq 0$. We want to show that

$$\sum_{\tilde{t}=t,t'} \sum_{\theta \in A} \mu(\theta)\sigma'(\tilde{t}|\theta)p_{\tilde{t}}(\alpha';\theta) - \sum_{\theta \in R} \mu(\theta)\sigma'(\tilde{t}|\theta)p_{\tilde{t}}(\alpha';\theta)$$
$$\geq \sum_{\tilde{t}=t,t'} \sum_{\theta \in A} \mu(\theta)\sigma'(\tilde{t}|\theta)p_{\tilde{t}}(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta)\sigma'(\tilde{t}|\theta)p_{\tilde{t}}(\alpha;\theta). \quad (9)$$

This holds because

$$\sum_{\tilde{t}=t,t'} \sum_{\theta \in A} \mu(\theta)\sigma(\tilde{t}|\theta)p_{\tilde{t}}(\alpha';\theta) - \sum_{\theta \in R} \mu(\theta)\sigma(\tilde{t}|\theta)p_{\tilde{t}}(\alpha';\theta)$$
$$\geq \sum_{\tilde{t}=t,t'} \sum_{\theta \in A} \mu(\theta)\sigma(\tilde{t}|\theta)p_{\tilde{t}}(\alpha;\theta) - \sum_{\theta \in R} \mu(\theta)\sigma(\tilde{t}|\theta)p_{\tilde{t}}(\alpha;\theta) \quad (10)$$

Moreover, because $\alpha'(x,t') \geq \alpha(x,t')$ and $\alpha'(x,t) \leq \alpha(x,t)$, we have,

$$-\sum_{\theta \in R} \mu(\theta)\delta(\theta)p_t(\alpha';\theta) \geq -\sum_{\theta \in R} \mu(\theta)\delta(\theta)p_t(\alpha;\theta), \quad (11)$$
$$\sum_{\theta \in R} \mu(\theta)\delta(\theta)p_{t'}(\alpha';\theta) \geq \sum_{\theta \in R} \mu(\theta)\delta(\theta)p_{t'}(\alpha;\theta). \quad (12)$$

Summing (10), (11) and (12), we get inequality (9). If (10) holds strictly, then (9) holds strictly as well.

Next, I show that $v(\alpha,\sigma)$ is quasi-supermodular in α: $v(\alpha,\sigma) \geq (>) v(\alpha \wedge \alpha',\sigma) \Rightarrow v(\alpha \vee \alpha',\sigma) \geq (>) v(\alpha',\sigma)$. This follows directly from the fact that $v(\alpha,\sigma)$ is linear in α and therefore $\frac{\partial^2 v(\alpha,\sigma)}{\partial\alpha(x,\tilde{t})\partial\alpha(x',\tilde{t}')} = 0$. The order $\geq_C$ is a variation of the usual order on $\mathbb{R}^{|X|} \times \mathbb{R}^{|T|}$ and therefore $v(\alpha,\sigma)$ is supermodular (Topkis, 1978). This implies it is quasi-supermodular.

We now claim that $-v(\alpha,\sigma)$ has the single-crossing property in $(\sigma;\alpha)$ and is quasi-supermodular in σ. The proof of single-crossing property works exactly in the same way as for the single crossing property of $v(\alpha,\sigma)$ in $(\alpha;\sigma)$. We can set write $\alpha(x,\tilde{t}) = \alpha'(x,\tilde{t}) + \delta(x,\tilde{t})$ for $\tilde{t} = t,t'$ and follow the same steps. Quasi-supermodularity follows from the linearity of $-v(\alpha,\sigma)$.

We have shown that the game satisfies the condition in Milgrom and Shannon (1994) for equilibrium existence: (1) each player's strategy space is a compact lattice, (2) the payoff function is continuous in (α,σ) and (3) the payoff function is quasi-supermodular in the own strategy and satisfies the single crossing properties just shown. Thus an equilibrium exists.

In this zero-sum game, it implies that $\max_{\alpha\in C}\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R) = \min_{\sigma^R}\max_{\alpha\in C} v(\alpha,\sigma^A,\sigma^R)$.

$\square$

D Generalisations of Proposition 1 to Many Signals

In this section, I show how the results of Proposition 1 generalise naturally when the DM (1) there is an order on types such that the dominant test satisfies MLRP and (2) the DM uses an increasing strategy. Throughout this section, I assume that $X \subset \mathbb{R}$.

A test t has an MLRP-order if there is an order $\geq_M$ on Θ such that $\theta \geq_M \theta'$ if and only if for any $x > x'$,

$$\pi_t(x|\theta)\cdot\pi_t(x'|\theta') \geq \pi_t(x'|\theta)\cdot\pi_t(x|\theta').$$

The DM's strategy is *increasing* if $x' > x \Rightarrow \alpha(x',t) \geq \alpha(x,t)$ for all $t\in T$. Let I be the set of increasing strategies. A special type of increasing strategies are *cutoff strategies*: for any $t\in T$, $\alpha(x,t) > 0 \Rightarrow \alpha(x',t) = 1$ for any $x' > x$. Denote by C the set of cutoff strategies.

It is worth noting that in this environment, it might not be optimal to use the dominant test as an increasing strategy might not be adapted to it. However, the optimal menu contains the dominant test if the unconstrained optimal strategy associated with the dominant test is in fact increasing.

The following result generalises the idea that conditions like (U-2) or (U-3) imply that having a dominated test in the menu is optimal.

Denote by $F_t(x|\theta) = \sum_{x'\leq x}\pi_t(x'|\theta)$ the cdf of $\pi_t(\cdot|\theta)$.

Proposition 9. *Let $X \subset \mathbb{R}$. Suppose there is a test $t \succeq t'$ for all $t' \in T$ and that t is MLRP-ordered with order $\geq_M$. Suppose also that the DM must use an increasing strategy.*

Take $\alpha \in \arg\max_{\alpha'\in I}\sum_{\theta\in A}\mu(\theta)p_t(\alpha';\theta) - \sum_{\theta\in R}\mu(\theta)p_t(\alpha';\theta)$ and let $x^* = \max\{x : \alpha(x,t) = 0\}$.

If there is $\underline{\theta}$ and $\theta\in A$ with $\theta \leq_M \underline{\theta}$ such that

$$\sum_{\theta\in A,\theta\leq_M\underline{\theta}}\mu(\theta)F_t(x^*|\theta) - \sum_{\theta\in R,\theta\leq_M\underline{\theta}}\mu(\theta)F_t(x^*|\theta) \geq F_t(x^*|\underline{\theta})\left(\sum_{\theta\in A,\theta\leq_M\underline{\theta}}\mu(\theta) - \sum_{\theta\in R,\theta\leq_M\underline{\theta}}\mu(\theta)\right), \tag{U'}$$

then there is an optimal menu containing some $t' \preceq t$.

If condition (U') holds for all $x \in X$, then it is always true that an optimal menu containing some $t' \preceq t$ is optimal. Under binary signals, condition (U') is a version of condition 'Imprecise at the bottom' (U-2) in Proposition 1.

Note also that under the framework of Section 5 with effort and communication, all strategies must be increasing for the agent to put effort. Therefore, condition (U') and thus (U-2) in the binary signal case imply that an optimal menu contains a dominated test.

The intuition for (U') is the same as 'Imprecise at the bottom' (U-2). When using a cutoff strategy, the probability of being acepted is $p_t(\alpha;\theta) = 1 - F_t(x^*|\theta)$, i.e., the probability of sending a signal higher than the cutoff x^*. Condition (U') is thus also a form of 'imprecision at the bottom': it states that the DM prefers types below $\underline{\theta}$ to be pooled and all have distribution over signals $\pi_t(\cdot|\underline{\theta})$. If there is an A-type θ such that $\theta \leq_M \theta'$ for all $\theta' \in R$, then condition (U') is satisfied.

Proof. This follows from two observations. First, as argued in the discussion of Theorem 1, as the set of increasing strategies I is closed and convex, Theorem 1 also holds when restricting attention to that class of strategies.

Suppose the DM only offers t in the optimal menu and take $\alpha \in \arg\max_{\alpha' \in I} \sum_{\theta \in A} \mu(\theta) p_t(\alpha';\theta) - \sum_{\theta \in R} \mu(\theta) p_t(\alpha';\theta)$. The set of step functions: $\alpha(x,t) = \mathbb{1}[x > \tilde{x}]$ for some $\tilde{x} \in X \cup \{\min X - 1\}$ is the set of extreme points of the set of increasing functions (see e.g., Lemma 2.7 in Börgers, 2015).[26] Because the DM's payoffs are linear in α when offering only one test, it is without loss of optimality to take the strategy associated as being a step function with step at x^* as defined in the proposition.

Because condition (U') is satisfied, there is $\underline{\theta}$ and $\theta \in A$ with $\theta \leq_M \underline{\theta}$ such that

$$\sum_{\theta \in A, \theta \leq_M \underline{\theta}} \mu(\theta) \left(F_t(x^*|\theta) - F_t(x^*|\underline{\theta}) \right) \geq \sum_{\theta \in R, \theta \leq_M \underline{\theta}} \mu(\theta) \left(F_t(x^*|\theta) - F_t(x^*|\underline{\theta}) \right).$$

Take any $t' \in T$ and set $\alpha'(x,t') = 1 - F_t(x^*|\underline{\theta})$ and $\alpha'(\cdot,t) = \alpha(\cdot,t)$. Because types are MLRP-ordered, we have that $F_t(x^*|\theta) \geq F_t(x^*|\theta')$ for all $\theta' >_M \theta$. Therefore, only types

[26] Setting $\tilde{x} = \min X - 1$ allows the DM to always accept.

$\theta \leq_M \underline{\theta}$ prefer to deviate to t'. By condition (U'), we get

$$\sum_{\theta \in A, \theta \leq_M \underline{\theta}} \mu(\theta) p_{t'}(\alpha'; \theta) - \sum_{\theta \in R, \theta \leq_M \underline{\theta}} \mu(\theta) p_{t'}(\alpha'; \theta) \geq \sum_{\theta \in A, \theta \leq_M \underline{\theta}} \mu(\theta) p_t(\alpha'; \theta) - \sum_{\theta \in R, \theta \leq_M \underline{\theta}} \mu(\theta) p_t(\alpha'; \theta).$$

Therefore, α' is a profitable deviation in the max-min problem or another maximum.

□

We can also combine the ideas of Proposition 2 and Proposition 1 to show that when the dominant test t is single-peaked with respect to $\geq_M$, i.e., there is $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq_M \theta \leq_M \theta_2\}$, all equilibria where t is used are payoff equivalent to equilibria where only t is used. This is the equivalent to showing that if condition (U-1) is not satisfied, a singleton menu containing t dominates any other menu containing t and other tests. Here we require the DM to use a cutoff strategy.

Proposition 10. *Let $X \subset \mathbb{R}$. Suppose there is a test $t \succeq t'$ for all $t' \in T$ and that all $t' \in T$ are MLRP-ordered with order $\geq_M$. Suppose also that the DM must use a cutoff strategy.*

If there is $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq_M \theta \leq_M \theta_2\}$, the menu $\{t\}$ is optimal among the menus containing t.

Proof. The proof proceeds similarly as the proof to show that if (U-1) is not satisfied, all equilibria having t on path have the same payoffs as having all types choosing t.

Let α be the strategy used by the DM in some equilibrium. As in the proof of Proposition 2, if $t \succeq t'$ and the DM uses a cutoff strategy, $p_t(\alpha; \theta) - p_{t'}(\alpha; \theta)$ is single-crossing in θ. Suppose there is some $\theta, \theta' \in A$ choosing t, t' respectively. Then the set of types choosing t is such that there is a cutoff type θ^* where $\theta \leq_M \theta^* \Rightarrow \theta \in A$ and $\theta >_M \theta^* \Rightarrow \theta \in R$. Let $\tilde{\geq}_M$ be the order such that $\theta \geq_M \theta' \Leftrightarrow \theta' \tilde{\geq}_M \theta$. Then the strategy $\alpha(\cdot, t)$ is decreasing with respect to $\tilde{\geq}_M$ and all types choosing t have the property $\theta \in A, \theta' \in R \Rightarrow \theta \tilde{\geq}_M \theta'$. If $\alpha(\cdot, t)$ is not constant, the DM would be strictly better off using a strategy $\alpha'(x, t) = \alpha^* \in [0, 1]$ for some constant α^*, see e.g., Lemma 3.3 in Chi (2014) or the proof of Lemma 3 in Quah and Strulovici (2009). Let $\alpha_0(x, t) = 0$ and $\alpha_1(x, t) = 1$ for all $x \in X$. The strategy $\alpha'(\cdot, t)$ is a convex combination of $\alpha_1(\cdot, t)$ and $\alpha_0(\cdot, t)$, two cutoff strategies. By the linearity of the preferences in α, these would be a profitable situation. Therefore, it must be that either $\alpha(\cdot, t) = 0$ or $= 1$. In that case, either all types choose t or none of them. □

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